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An algebraic slice in the coadjoint space of the Borel and the Coxeter element

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Abstract

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{b}$ a Borel subalgebra. The algebra Y of polynomial semi-invariants on the dual $\mathfrak{b}^*$ of $\mathfrak{b}$ is a polynomial algebra on rank $\mathfrak{g}$ generators (Grothendieck and Dieudonné (1965–1967)) [16]. The analogy with the semisimple case suggests there exists an algebraic slice to coadjoint action, that is an affine translate $y + V$ of a vector subspace of $\mathfrak{b}^*$ such that the restriction map induces an isomorphism of Y onto the algebra $R[y + V]$ of regular functions on $y + V$. This holds in type A and even extends to all biparabolic subalgebras (Joseph (2007)) [20]; but the construction fails in general even with respect to the Borel. Moreover already in type $C(2)$ no algebraic slice exists.

Very surprisingly the exception of type $C(2)$ is itself an exception. Indeed an algebraic slice for the coadjoint action of the Borel subalgebra is constructed for all simple Lie algebras except those of types $B(2m)$, $C(n)$ and $F(4)$.

Outside type A, the slice obtained meets an open dense subset of regular orbits, even though the special point y of the slice is not itself regular. This explains the failure of our previous construction.

MSC: 17B35

Keywords: Invariants; Adapted pairs

0. Preamble

Let $\mathfrak{g}$ be a simple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and set $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. Let P^+ denote the corresponding set of dominant weights. There is a canonically determined

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ideal $\mathfrak{b}_E$ of $\mathfrak{b}$ containing $\mathfrak{n}$ such that the algebra generated by semi-invariants on $\mathfrak{b}^*$ coincides with the algebra $Y(\mathfrak{b}_E)$ of invariants on $\mathfrak{b}_E^*$. Moreover the latter algebra is polynomial on rank $\mathfrak{g}$ generators [18]. The analogy with the semisimple case suggests there exists an algebraic slice to coadjoint action defined to be an affine translate $y + V \subset \mathfrak{b}_E^*$ of a vector subspace V of $\mathfrak{b}_E^*$ such that the restriction map induces an isomorphism of $Y(\mathfrak{b}_E)$ onto the algebra $R[y + V]$ of regular functions on $y + V$. This holds in type A and even extends to all biparabolic subalgebras [22]; but the construction fails in general even with respect to the Borel. Moreover already in type C_2 no algebraic slice exists.

An attempt [13,14] to construct $Y(\mathfrak{g})$ via the Hopf dual of $U(\mathfrak{g})$ led to the subalgebra

$$A(\mathfrak{g}) := \bigoplus_{\lambda \in P^+} Y\big(\mathfrak{n}^-\big)_{-\lambda} Y(\mathfrak{b}_E)_\lambda,$$

of $S(\mathfrak{g})$, with the goal being to show that $((\operatorname{ad} U(\mathfrak{g}))A(\mathfrak{g}))^{\mathfrak{g}} = Y(\mathfrak{g})$. This is true in types A and C by [13].

A main result of this work is that there exists a regular nilpotent element $y \in \mathfrak{g}$ and a subspace V of $\mathfrak{g}$ of dimension rank $\mathfrak{g}$ such that the restriction map induces an algebra isomorphism of $A(\mathfrak{g})$ onto $R[y + V]$. From a purely algebraic point of view this is remarkably similar to an algebraic slice for $\mathfrak{b}_E$. Indeed it would itself yield such a slice if restricted to $y + V$ the non-zero elements in the one-dimensional spaces $Y(\mathfrak{n}^-)_{-\lambda}$ all became non-zero scalars. Unlike the case of $\mathfrak{g}$ simple, the choice of the regular nilpotent element y is rather delicate. Roughly speaking it is given by the "square root" of the unique longest element w_0 of the Weyl group acting on the standard regular ad-nilpotent element which is the sum of simple root vectors. For example when $w_0 = -Id$ and the Coxeter number is divisible by 4, say equals $4n$, this element is the nth power of a carefully chosen Coxeter element and has square equal to w_0.

Very surprisingly the exception of type C_2 mentioned above is itself an exception. Thus it is possible to make use of the above result to obtain an algebraic slice for the coadjoint action of the Borel subalgebra of all simple Lie algebras except those of types B_{2m}, C_n, F_4. This is achieved by "switching" which modifies the choice of $y + V$. The slice obtained meets an open dense subset of regular orbits, even though the special point y of the slice is not itself regular, outside type A. This explains the failure of our previous construction.

In principle the above construction can be extended to give a more satisfactory understanding of algebraic slices of biparabolics in type A, for which the construction of y was rather ad hoc and in the appropriate cases for other types. (Such slices generally tend to exist when the Cartan subalgebra of the truncated biparabolic is sufficiently large.) More generally it is hoped that it may provide algebraic slices for truncated biparabolics at least when the invariant algebra is polynomial, which is frequently the case [13,20].

1. Introduction

1.1. Let $\mathfrak{a}$ be an algebraic Lie algebra and denote by the corresponding capital Roman letter in this case A its adjoint group acting on $\mathfrak{a}^*$ by transposition. Let $S(\mathfrak{a})$ be the symmetric algebra over $\mathfrak{a}$ and $Y(\mathfrak{a})$ the subalgebra of A invariant elements, which we identify with the A invariant polynomial functions on $\mathfrak{a}^*$.

1.2. The coadjoint action on $\mathfrak{a}^*$ is said to admit an affine slice if there exists an affine translate $y + V \subset \mathfrak{a}^*$ of a vector subspace V of $\mathfrak{a}^*$ which roughly speaking meets almost every orbit exactly once. We give a precise formulation in Section 7 and compare it to the notion of a slice in the sense of Luna [28] and Slowody [38], which turns out to be rather different.

Our notion of an affine slice is rather special. Assume that $S(\mathfrak{a})$ admits no proper semi-invariants. We show in 7.7, using a result of Hinich (12.3), that the restriction of functions defines an embedding of $Y(\mathfrak{a})$ into the algebra $R[y+V]$ of regular functions on $y+V$, which induces an isomorphism of fields of fractions. In particular Fract $S(\mathfrak{a})^A$ must be pure (see Lemma 7.7 and Section 7.11).

1.3. We say that $\mathfrak{a}^*$ admits an algebraic slice if there exists an affine translate $y+V \subset \mathfrak{a}^*$ of a vector subspace V of $\mathfrak{a}^*$ such that restriction of functions induces an isomorphism of $Y(\mathfrak{a})$ onto $R[y+V]$. This of course implies that $Y(\mathfrak{a})$ is a polynomial algebra, which is much more special. In 7.7 we show an algebraic slice admits an affine dense subset which is an affine slice in the sense of 1.2.

Our present notion of an algebraic slice is equivalent to what is also called a Weierstrass section in [34, 2.2] for $\mathfrak{a}$ reductive – see 7.1.

In the situations we encounter here the construction of an affine slice takes only a few lines (10.1) whilst the construction of an algebraic slice occupies almost the entire paper, besides requiring most of [18, Sect. 4].

1.4. Let $\mathfrak{g}$ be a simple Lie algebra. It was already considered a remarkable achievement of Chevalley [6] to show that $Y(\mathfrak{g})$ is polynomial. Further to this Kostant [27, Thm. 0.10] proved that in this case coadjoint action admits an algebraic slice with the additional property that it exactly meets every regular orbit. However this last refinement results from the fact that the nilfibre of the categorical quotient map $\mathfrak{g}^* \to g^*//G$ is irreducible. Moreover the construction further requires that the nilfibre admits a regular orbit. These two conditions will seldom be satisfied in the situations we consider.

1.5. The results of Chevalley and Kostant used the special properties of $\mathfrak{g}$ being simple and in this the Weyl group W played a major role. Indeed the latter and reductivity were central to Chevalley's proof of polynomiality. Moreover the use of W provided some information on the degree of these polynomial generators. In particular their sum was found to be given by $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{index} \mathfrak{g})$, a result central to Kostant's theorem. Perhaps for this reason it was not suspected for a long time that these results could possibly extend to other classes of Lie algebras.

1.6. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ be its nilradical. A remark in a paper by Schmidt [37] notes that Kostant had shown $Y(\mathfrak{n})$ to be polynomial generalizing an earlier result of Dixmier [8] for type A. In [18, 4.12] the present author recovered these results though it turned out by a quite different method. Furthermore in [18, 4.16] it was shown that the subalgebra $Sy(\mathfrak{b})$ of $S(\mathfrak{b})$ generated by the semi-invariants was also polynomial on exactly rank $\mathfrak{g}$ generators. Precisely when -1 does not belong to W, extra generators are obtained and their degrees were calculated.

1.7. Following the introduction by Drinfeld and Jimbo of quantum groups, it was found possible to study invariants for $\mathfrak{g}$ simple in a quantum setting. Surprisingly this was much easier and in [12, Thm. 1] polynomiality was established for the semi-invariant subalgebra for any parabolic subalgebra *including* $\mathfrak{g}$ *itself*. It transpired that the reason for this simplification had come from a duality (sometimes called Drinfeld duality, or the Rosso form) between a quantized enveloping algebra and its Hopf dual, whilst in the Hopf dual the algebraic version of the Peter–Weyl theorem easily allows one to describe invariants even for those coming from parabolic subalgebras. Now this last fact applied equally well to the Hopf dual of the classical enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Surprisingly this did not seem to have been noticed before except perhaps as it pertained to invariants for $\mathfrak{g}$ itself (in which case the invariants are traces) and for its Borel subalgebra (in which case the invariants are matrix coefficients between highest and lowest weight vectors of a given simple module). In the case of the Borel, this had been used by Kostant to obtain a description of $Y(\mathfrak{n})$. Of course Drinfeld duality disappears, but Kostant used the canonical filtration on $U(\mathfrak{g})$ to induce a filtration on its Hopf dual eventually providing a linear map [13, Sect. 6] to $S(\mathfrak{g})$. (Here we remark that for $\mathfrak{g}$ this construction does not seem of much use even though it that case the above linear map is bijective [13, 6.9].) We have extended this construction to the parabolic [13] and the biparabolic case [20].

The first glitch in the above construction is that unlike the quantum case one fails to obtain all the generators of $Y(\mathfrak{n})$ in this fashion except in types A and C [13, 6.9,7.2]. (In other words our linear map is not surjective unlike the quantum case.) The remaining generators turn out to be square roots of certain of those invariants which are obtained. More generally if $\mathfrak{g}$ is simple and $\mathfrak{p}$ is a proper parabolic subalgebra, then the weight spaces of $Sy(\mathfrak{p})$ are finite dimensional [13, 5.4.3]. Following this one can speak of upper and lower bounds on (the dimensions of the weight subspaces of) $Sy(\mathfrak{p})$. Then it is possible to use the construction of invariants in the Hopf dual combined with the above filtration to obtain [13, Sect. 6] a lower bound on $Sy(\mathfrak{p})$. Quite remarkably this coincides in most cases with an upper bound (obtained by a quite different method, [13, Sect. 4]) and then the semi-invariant algebra is found to be polynomial [13, 7.2]. However when these bounds fail to coincide, the description of the missing semi-invariants is more complicated and as yet incomplete [13, 7.2]. In small rank cases it was checked that these algebras are polynomial (and moreover one can sometimes improve the upper bound [23, 6.11]); but in type E_8, Yakimova [39] has shown that polynomiality can fail for the "Heisenberg" parabolic defined as the normalizer of the highest root vector. In this case it is just in type E_8, when our lower and improved upper bounds fail to coincide [23, 6.14]. In general it is not even known if $Sy(\mathfrak{p})$ is finitely generated.

1.8. The polynomiality of $Sy(\mathfrak{p})$ means that we can ask if coadjoint action admits an algebraic slice. Actually it is more appropriate to cut down (to a canonically determined) ideal $\mathfrak{p}_E$ of $\mathfrak{p}$ for which $Sy(\mathfrak{p}) = Y(\mathfrak{p}_E)$, that is to say semi-invariants become invariants in the smaller algebra (and none disappear). We call $\mathfrak{p}_E$ a truncated parabolic subalgebra. In type A, it was shown that an algebraic slice exists for all truncated parabolics and indeed for all truncated biparabolics [21,22]. We remark that truncation is a fairly trivial procedure which just removes part of the Cartan subalgebra. However if it requires the removal of all the Cartan subalgebra (for example in the case of the Borel when $-1 \in W$) then there is no hope of constructing an adapted pair (see below).

The second glitch in the above is that outside type A an algebraic slice generally fails to exist even in the Borel case, for example in type C_2. This appeared to be related to another difficulty. Indeed we found in [14, Sect. 6] that the sum of the degrees of the generators of $Sy(\mathfrak{b})$ is bounded by $\frac{1}{2}(\dim \mathfrak{b} + \operatorname{index} \mathfrak{b})$; but this bound can be strict. More generally let $\mathfrak{a}$ be a finite dimensional Lie algebra. Call an element $y \in \mathfrak{a}^*$ regular if it generates an A orbit of maximal dimension and let $\mathfrak{a}^*_{reg}$ be the set of all regular elements in $\mathfrak{a}^*$. It is an open dense subset of $\mathfrak{a}^*$ and consequently has codim $\geqslant 1$. Call $\mathfrak{a}$ singular if equality holds. It was shown recently by Ooms and Van den Bergh [31, Cor. 1.5], that if $Sy(\mathfrak{a})$ is polynomial, then the degrees of its homogeneous generators have sum $\leqslant \frac{1}{2}(\dim \mathfrak{a} + \operatorname{index} \mathfrak{a})$ with equality if and only if $\mathfrak{a}$ is non-singular, assuming here and in the remainder of the introduction that $Sy(\mathfrak{a}) = Y(\mathfrak{a})$. This as noted in [31, Sect. 3] can always be assumed by cutting down $\mathfrak{a}$ to an ideal, though not necessarily in a canonical fashion (unless $\mathfrak{a}$ is almost algebraic, see [14, Sect. 8] for example, where we reproduced a calculation due to W. Borho and independently to C. Chevalley).

Our method [22, 2.7] to produce an algebraic slice requires the construction of an adapted pair (h, y) consisting of a *regular* element y of $\mathfrak{a}^*$ and an ad-semisimple element $h \in \mathfrak{a}$ such that $(\operatorname{ad} h)$ $y = -y$. Here two further technical conditions were imposed but this turns out to be unnecessary (see 7.13). In this case the slice is just $y + V$, where V is any ad h stable complement to $(\operatorname{ad} \mathfrak{a})y$ in $\mathfrak{a}^*$. Now if $\mathfrak{a}$ is singular, then an irreducible component $\mathscr{C}$ of $\mathfrak{a}^* - \mathfrak{a}^*_{reg}$ of codimension 1 must be the zero set of an element which is semi-invariant and so under the present hypothesis must belong to $Y(\mathfrak{a})$. In particular $\mathscr{C}$ contains the nilfibre $\mathscr{N}$ of the categorical quotient map, whilst the relation $(\operatorname{ad} h)y = -y$ forces $y \in \mathscr{N} \subset \mathscr{C}$ contradicting its regularity.

In particular if $\mathfrak{a}$ is singular, $\mathscr{N}$ has no regular elements. This can also happen outside $\mathfrak{a}$ singular, for example if $\mathfrak{a}$ is the truncated Borel $\mathfrak{b}_E$ in type C_2, which is non-singular. (It can even happen that the nilfibre admits irreducible components some having and some not having regular elements [22, 11.3].) This circumstance can and does sometimes exclude the existence of an algebraic slice. However what we find here (9.4) is that one can often obtain an algebraic slice $y + V$ with the special point y not being regular, though not when $\mathfrak{a}$ is non-singular. Previously it had not been at all obvious how such a slice should be constructed.

Related to the above we remark that if an algebraic slice (alias a Weierstrass section in Popov's terminology [34, 2.2]) exists for a reductive Lie algebra $\mathfrak{a}$ acting linearly on a finite dimensional vector space X such that the set X_s of non-regular elements in X form a subvariety of codimension > 1, then a result of Popov [34, Thm. 2.2.15] implies that the special element y of $y + V$ (and hence by 7.8 all elements) must be regular. This would explain the observations of the previous paragraph except that the result of Popov is only announced for $\mathfrak{a}$ reductive. Besides in Popov's work there is no general recipe for finding an adapted pair which can be very difficult in the case when $\mathfrak{a}$ is not reductive (see [22] for example), nor a general recipe for finding an algebraic slice/Weierstrass section if X_s has codimension 1. However Popov does give an example [34, 2.2.16] of a Weierstrass section $y + V$ in the case when X_s has codimension 1. This is for $\mathfrak{a} = \mathfrak{sl}(n)$ acting in $X := Mat_{n\otimes n}$ by left multiplication. Popov notes that in this case X_s is the zero set of the determinant. Moreover y is the identity matrix deprived of its first entry, whilst V is the vector subspace of $Mat_{n\otimes n}$ formed spanned by this first entry, so only y is not regular in $y + V$. (Compare this to Example 3 of 11.4, where the non-regular elements in $y + V$ form a closed subvariety of dimension 1.)

In the above general situation of an algebraic group A acting linearly on a vector space X, the "singular" set X_s is A stable and so is its codimension 1 component. By Krull's theorem this is the zero set of a polynomial on X which is necessarily semi-invariant. In the case when $\mathfrak{a}$ is a (truncated) Borel subalgebra $\mathfrak{b}_E$ and X is its dual space, we show (Proposition 13.3) that this semi-invariant is exactly the product of the "missing invariant generators" The proof uses arguments in [25] and had been presented in a seminar in Chevaleret, Paris in June 2010. The interest in this result lies in the fact that it gives "an explanation" of why these "missing generators" must appear. Indeed it is because their product is exactly the fundamental semi-invariant (in the sense used by Ooms and Van den Bergh [31]) of the truncated Borel. (Unfortunately our proof is not the best imaginable (see 13.4) because it uses the existence of these invariants. It is therefore not easy to transfer it to the biparabolic situation which is anyway somewhat more complex.)

Thus the present paper gives two reasons why these "missing generators" are "needed", namely the one above and the one noted in 1.10 below.

1.9. That $Sy(\mathfrak{p})$ has finite dimensional weight subspaces obviously breaks down if $\mathfrak{p}$ is $\mathfrak{g}$ itself. Yet because the $\mathfrak{g}$ is reductive the construction of $Y(\mathfrak{g})$ using the Hopf dual and the above filtration gives the whole [13, 6.9] of $Y(\mathfrak{g})$. Nevertheless because we have lost finite dimensionality of weight subspaces, we cannot thereby deduce, unlike the quantum case [19, 7.1.17(iii)], too much about the structure of $Y(\mathfrak{g})$, for example that it is polynomial. On the other hand the fact that $Sy(\mathfrak{b})$ is also a polynomial algebra on rank $\mathfrak{g}$ generators leads one to consider the possibility of there being a natural map taking $Sy(\mathfrak{b})$ to $Y(\mathfrak{g})$. One way to do this is through a certain fattening out of $Sy(\mathfrak{b})$ into a larger subalgebra $A(\mathfrak{g})$ of $S(\mathfrak{g})$, which is still polynomial on rank $\mathfrak{g}$ generators and specializes to $Sy(\mathfrak{b})$. Then one takes G invariants in $\operatorname{ad} U(\mathfrak{g})A(\mathfrak{g})$. In this fashion one does obtain a natural map which is linear but not necessarily an algebra homomorphism. We had conjectured [14, 4.10–4.13], this map to be surjective, equivalently that $(\operatorname{ad} U(\mathfrak{g})A(\mathfrak{g}))^G = Y(\mathfrak{g})$. To this date we have just shown it to be bijective in types A and C, [14, 4.8].

A third glitch is that the above map is not injective outside types A and C. Here we remark that there a growing number of results in invariant theory which hold in types A and C but fail for an arbitrary simple Lie algebra (for a partial list see [22, 1.4]). Indeed it is even rather remarkable to find results in invariant theory which holds for all simple Lie algebras.

1.10. A main result of our present paper is to show that there exists an affine translate $y + V \subset \mathfrak{g}^*$ such that the restriction of $A(\mathfrak{g})$ induces an isomorphism of $A(\mathfrak{g})$ onto $R[y + V]$. We call this a metaslice. Under this reformulation the first two glitches disappear and notably there is no distinction between types A, C and the rest. We emphasize that this is not at all a mere juggling of notions but is a result which depends crucially on several remarkable coincidences which hold for *all* $\mathfrak{g}$ simple. In particular the "missing invariant generators are needed" (see Remark 1 of 5.12). Notably we obtain a general description (via Theorems 3.2, 5.12) of degrees of generators in terms of some Weyl group combinatorics involving notably a particular Coxeter element. In this our goal had been to prove the conjecture noted in 1.9 above; but so far we are unable to see if our metaslice has any bearing on the matter (cf. Remark 8.9). Thus for the moment the consequences of the third glitch remain. However what one does find (Proposition 8.9) is that G saturation set of our metaslice $y + V$ is not dense in $\mathfrak{g}^*$ exactly when $\mathfrak{g}$ is not of type A or C.

1.11. Somewhat surprisingly we were able to find processes of "switching" and of "exotic switching" which allows a metaslice to be morphed into an algebraic slice for the truncated Borel subalgebra of $\mathfrak{g}$, outside types B_{2m}, C_n, F_4. Here we note that in type C_2 no algebraic slice can exist, whilst the remaining exceptions above are exactly those for which the Kostant cascade $\mathscr{K}$, see (2.15), admits a hereditary subset of type C_2. On the other hand (if $-1 \in W$) then the direct sum V of the root vectors $x_{-\beta}$: $\beta \in \mathscr{K}$ provides an affine slice being an open subset of V on which in particular all the non-minimal elements of $\mathscr{K}$ can be inverted. Finally in types B_{2m}, F_4 we use exotic switching to show that a better result is possible, namely it is only necessary to invert the highest root vector in an open subset of an appropriate affine translate of a vector subspace of $\mathfrak{g}^*$, see 10.2, 10.3.

A longer term goal is to extend these results to most if not all biparabolics $\mathfrak{p}$ and in particular to obtain a description of the generators of $Sy(\mathfrak{p})$ which would involve the Weyl group. In this Yakimova's counter-example [39], will have to be better understood. For the moment we just remark that (Laurent) polynomiality is recovered at the expense of localization at the highest root vector, as was already clear from the analysis in [18, Sect. 4].

1.12. Following a question of R. Tange we would like to say why we considered the algebra $A(\mathfrak{g})$ in our conjectured presentation of $Y(\mathfrak{g})$. The answer is very simple. In some admittedly rather convoluted sense, $A(\mathfrak{g})$ contains the set of leading order terms of $Y(\mathfrak{q})$ for any truncated parabolic $\mathfrak{q}$ [12, 4.2.8] or biparabolic [20, 6.6] of a simple Lie algebra $\mathfrak{g}$. Then for example taking $\mathfrak{q}$ to be $\mathfrak{g}$ itself one would expect to be able recover $Y(\mathfrak{g})$ as $(\operatorname{ad} U(\mathfrak{g})A(\mathfrak{g}))^{\mathfrak{g}}$. This argument does not quite work and we eliminated all reference to this question from [12] because the referee had commented that "Ca ressemble à une plaisanterie de déduire la structure de $S(\mathfrak{g})^{\mathfrak{g}}$ de celle de $S(\mathfrak{b})^{[\mathfrak{b},\mathfrak{b}]}$!". Still "One man's joke is another man's bible", to paraphrase a well-known English saying, so we persisted in our foolishness with the result published in [13, Thm. 4.9] and those announced in the present paper. Here we might even have taken the further step from folly to total madness by conjecturing $S(\mathfrak{g})$ to be generated by adjoint action on $\bigoplus_{\lambda,\mu\in P^+} Y(\mathfrak{n}^-)_{-\mu}Y(\mathfrak{b}_E)_\lambda$, from which our conjecture concerning $Y(\mathfrak{g})$ would follow. The search for a geometric interpretation to these conjectures, however misguided, led to the present work.

2. Remarks on the Coxeter element

2.1. Let us fix some fairly standard notation. Let $\mathfrak{g} = \mathfrak{n} + \mathfrak{h} + \mathfrak{n}^-$, be a triangular decomposition of a complex simple Lie algebra. Set $\ell = \dim \mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the corresponding set of all non-zero roots and $\pi \subset \mathfrak{h}^*$ (resp. $\pi^\vee \subset \mathfrak{h}$) the corresponding set of simple roots (resp. coroots). Set $\Delta^+ = \Delta \cup \mathbb{N}\pi$ and $\Delta^- = -\Delta^+$. We shall write $\pi = \{\alpha_i \colon i \in I\}$, adopting the Bourbaki [5] labelling. We borrow a term from Chemistry to define the valency of a simple root to be the number of its nearest neighbours in the Dynkin diagram. Thus an end root is univalent, whilst in types D and E there is a unique trivalent root. Let P (resp. P^+) denote the set of integral (resp. integral and dominant) weights. For each $\mu \in P^+$, let $V(\mu)$ denote the finite dimensional $U(\mathfrak{g})$ module with highest weight μ and if ν is a weight of $V(\mu)$, let v_ν, denote a vector of weight ν which of course may be far from unique. Let s_i denote the simple reflection defined by $i \in I$ and W the group these elements generate. Call a weight of $V(\mu)$ extremal if it lies in $W\mu$. Such weights have multiplicity one. Moreover for each $\mu \in P$ one can speak of the simple finite dimensional module with extremal weight μ. For each $\alpha \in \Delta$, let x_α denote the element of a Chevalley basis for $\mathfrak{g}$ of weight α.

We assume throughout this paper that $|\pi| > 1$.

2.2. Since the Dynkin diagram of π has no cycles there exists a decomposition of π as a disjoint union of subsets π_a, π_b so that the corresponding simple reflections in either of the subsets commute. Let I_a (resp. I_b) be the subset of I indexing π_a (resp. π_b). Since the Dynkin diagram is also connected, this decomposition is unique up to permutation. Let σ_a, σ_b denote the corresponding products of simple refections (which we can take in any order) and set $\sigma = \sigma_a\sigma_b$.

A Coxeter element is defined to be any conjugate of a product of all simple reflections taken once and in any order. Because π has no cycles it is rather immediate that they are all conjugate to σ above. However as we shall see the particular choice we have made has some extra properties with respect to reduced decomposition basically because it comes with a two generated Coxeter group $\widehat{C} := \langle \sigma_a, \sigma_b \rangle$. The latter is always a dihedral group whose order is twice the order c of σ. One calls c the Coxeter number. Its value is well known.

We shall verify the following

Lemma. *An element of $\widehat{C}$ is reduced in W if it is reduced in $\widehat{C}$.*

2.3. The above result is not entirely innocent. Indeed suppose the Coxeter number c is even, that is $c = 2m$. Then it is immediate from the above lemma that σ^m is the unique longest element w_0 of W. This generally fails for a Coxeter element even written as a product of ℓ distinct simple reflections. Again from the lemma it is immediate that $\operatorname{card}\Delta = \ell c$, a result which had been obtained from combined work of Coleman [6] and Kostant [26, Thm. 8.1, Cor. 8.2].

2.4. We give a proof of Lemma 2.2 by extending slightly a construction in [24, Sects. 2, 6.5, 8.2]. It is rather simple.

2.5. Let x be a positive real number and $\mathscr{A}$ the two by two Cartan matrix with 2 as diagonal entries and $-x$ as off-diagonal entries. As usual we realize $\mathscr{A}$ through a set π_c of simple roots $\{a, b\}$ and a set $\pi_c^\vee$ of simple coroots $\{a^\vee, b^\vee\}$ satisfying $a^\vee(a) = 2$, $a^\vee(b) = -x$ and the corresponding relations with a and b interchanged. Define reflections $\mathbf{r}_a, \mathbf{r}_b$ on the $\mathbb{R}$ vector space $\mathfrak{h}_c^*$ spanned by a, b, by the usual formula

$$\mathbf{r}_a\lambda = \lambda - a^\vee(\lambda)a,$$

and let C be the (Coxeter) group they generate. As noted in [24, Sect. 2.2], this is a finite group (precisely the dihedral group $\mathbb{Z}_n \ltimes \mathbb{Z}_2$) when $x = 2\cos\frac{\pi}{n}$. We may realize the simple roots a, b in the real plane as vectors of equal length and by convention we take b to be obtained from a through a counterclockwise rotation by $\pi - \pi/n$. For this reason, but more crucially because of the choices made in 2.15, a and b will *not* be interchangeable even though the Cartan matrix $\mathscr{A}$ is symmetric.

Let us recall some further easy facts.

Define the polynomials $P_n(x)$ by

$$P_{n+1} = xP_n - P_{n-1}, \quad \forall n \geqslant 0,$$

with the initial condition $P_{-1} = 0$, $P_0 = 1$. Up to a rescaling of the indeterminate x by a factor of 2, these are the Chebyshev polynomials of the second kind. One may check that

$$(\sin\theta)P_n(2\cos\theta) = \sin(n+1)\theta, \quad \forall n \in \mathbb{N}. \tag{$*$}$$

Now take $x = 2\cos\frac{\pi}{n}$ and set $p_i := P_i(x)$. As noted in say [24, 8.7] already from just elementary Euclidean geometry the $p_i : i = 1, 2, \ldots, n-1$ are chord lengths in the regular n-gon, so in particular are positive real numbers satisfying

$$p_i = p_{n-i-1}. \tag{$**$}$$

On the other hand by $(*)$

$$p_{n-1} = 0, \quad p_{n+i-1} = -p_{i-1}\colon \quad i = 1, 2, \ldots, n, \quad p_{2n} = 1. \tag{$***$}$$

Set $\Delta_c = C\pi_c$. It follows from the above and the relations in [24, Sect. 2.2] that $\Delta_c = \Delta_c^+ \cup \Delta_c^-$ where Δ_c^+ (resp. Δ_c^-) lies in the set of non-negative (resp. non-positive) real linear combinations of the simple roots.

It is advisable for the next section to stress exactly what the above analysis proves. The polynomial equation $P_{n-1}(x) = 0$ for degree $n-1$ has by $(*)$ the $n-1$ solutions $x = 2\cos t\pi/n$: $t = 1, 2, \dots, n-1$. However this is not what we really need! Rather it is the following fact (obtained from [24, 8.7] which in turn resulted from just fitting triangles together!).

Lemma. *The polynomial equations*

$$P_{n-2-i}(x) = P_i(x): \quad i = 1, 2, \dots, n-2,$$

admit $x = 2\cos\frac{\pi}{n}$ *as a solution.*

2.6. Recall the notation of 2.1, 2.2. Given $i \in I$, let $N(i)$ denote its set of neighbours in the Dynkin diagram. Define g_i: $i = 1, 2, \dots, \ell$, by the formula

$$xg_i = -\sum_{j \in N(i)} \alpha_j^\vee(\alpha_i) g_j, \quad \forall i = 1, 2, \dots, \ell. \tag{$*$}$$

It is immediate that these equations imply a polynomial identity for x and that the g_i: $i = 1, 2, \dots, \ell$, are polynomials in x. More precisely we have the following

Lemma. *Eq.* $(*)$ *admits as a solution* $x = 2\cos\frac{\pi}{c}$ *for some positive integer* c. *Moreover in this the* g_i: $i = 1, 2, \dots, \ell$ *are positive real numbers. In types D and E the value of* g_i: $i \in I$ *takes its maximal at the unique trivalent root.*

Proof. Take an end root α_1 on the branch of the Dynkin diagram so that α_1 is joined to a maximal number of roots α_j: $j = 1, 2, \dots, t$, forming a Dynkin diagram of type A_t. Set $g_1 = 1$. It easily follows from $(*)$ that $g_j = P_{j-1}(x)$. In type A_ℓ there is one additional relation namely $xg_\ell = g_{\ell-1}$ yielding $P_{\ell-2}(x) = xP_{\ell-1}(x) = P_\ell(x) + P_{\ell-2}(x)$ or simply $P_\ell(x) = 0$. Then by Lemma 2.5, we obtain a solution with $x = 2\cos\frac{\pi}{(\ell+1)}$. Moreover the g_i are positive since by 2.5 they are chord lengths.

A similar calculation in types B_ℓ, C_ℓ yields $xP_{\ell-1}(x) = 2P_{\ell-2}(x)$ and so $P_\ell(x) = P_{\ell-2}(x)$. Then by Lemma 2.5, we obtain a solution with $x = 2\cos\frac{\pi}{2\ell}$. Moreover the g_i are chord lengths (except the last which is a chord length divided by x) and so are positive.

In type D_ℓ we obtain $g_\ell + g_{\ell-1} = P_{\ell-3}(x)$ and $xg_{\ell-1} = xg_\ell = P_{\ell-2}(x)$. Thus $xP_{\ell-3}(x) = 2P_{\ell-2}(x)$, which gives $P_{\ell-2} = P_{\ell-4}$. Thus by Lemma 2.5 this has a solution with $x = 2\cos\frac{\pi}{2(\ell-1)}$ and the g_i being positive.

In type E_ℓ one similarly obtains the equation

$$P_\ell(x) = P_{\ell-4}(x) + P_{\ell-6}(x). \tag{$**$}$$

For $\ell = 6$, this yields $P_8(x) = P_2(x)$ and so $x = 2\cos\frac{\pi}{12}$ is a solution. Taking $\ell = 7$ in $(**)$, multiplying by x^2 eventually gives $P_9(x) = P_7(x)$ and so $x = 2\cos\frac{\pi}{18}$ is a solution. For $\ell = 8$ a similar process repeated several times of multiplying by x^2 and substituting gives $P_{16}(x) = P_{12}(x)$ and so $x = 2\cos\frac{\pi}{30}$ is a solution. As before the g_i are positive since they are chord lengths divided by powers of x.

In type F_4, we obtain $g_2 = P_1(x)$, $2g_3 = P_2(x)$, $xg_3 = g_2 + g_4$ and $xg_4 = g_3$ which yields $P_4(x) = P_2(x) + P_0(x)$. This is the same solution as in type E_6.

For the last part let $1 \in I$, be the label of any univalent root and set $g_1 = 1$. Label the successive roots in a chain leading to the trivalent root by $g_2, g_3, \dots,$ until the unique trivalent root g_t is reached. Then as above $\{g_i = p_{i-1}\colon i = 1, 2, \dots, t\}$ and being chord lengths in the regular c-gon are strictly increasing given $t \leqslant [\frac{c+1}{2}]$. By the above calculation of c this last condition is satisfied in types D and E. $\square$

Remark. One may note that the computed values of c above coincides with the known values of the Coxeter number but this is obviated by Lemma 2.8, which needs only that the g_i are positive.

2.7. Eq. 2.6 $(*)$ has the following meaning which we state as an easily verified lemma. As in 2.2, let $\widehat{C}$ denote the subgroup of W generated by σ_a, σ_b.

Lemma. *There is a group homomorphism $\psi : \widehat{C} \to C$ defined by $\psi(\sigma_a) = \mathbf{r}_a$, $\psi(\sigma_b) = \mathbf{r}_b$ and a $\mathbb{Z}\widehat{C}$ linear map $\psi : \mathbb{Z}\pi \to \mathbb{Z}[x]a + \mathbb{Z}[x]b$ defined by $\psi(\alpha_i) = g_i a\colon \forall i \in I_a$, $\psi(\beta_i) = g_i b\colon \forall i \in I_b$.*

Remark. The positivity of the g_i and the definition of π_a, π_b imply that the image under ψ of a root γ is a multiple of a (resp. of b) exactly when $\gamma \in \pi_a$ (resp. $\gamma \in \pi_b$).

2.8. Set $\mathbf{r} = \mathbf{r}_a\mathbf{r}_b$ which is the Coxeter element of C. We remark that $\mathbf{r}$ is a rotation by $2\pi/c$ in an counterclockwise sense in virtue of our convention in 2.5. Observe that c in the conclusion of Lemma 2.6 is just the order of $\mathbf{r}$, by the formulae in [24, 2.2] or directly.

Lemma.

(i) *Suppose $c = 2m$. Then $\sigma^m = w_0$.*
(ii) *Suppose $c = 2m + 1$. Then $\sigma^m\sigma_a = \sigma_b\sigma^m = w_0$.*
(iii) *c is the Coxeter number, that is c is the order of σ.*
(iv) *The cardinality of any $\langle\sigma\rangle$ orbit in Δ is c. Moreover every orbit meets $\pi_a \sqcup -\pi_b$.*

Proof. (i) From the formulae in [24, 2.2], or directly, one checks that $\mathbf{r}^m = -1$. Now by Lemma 2.6 the $\mathbb{Z}$ linear map ψ takes Δ^+ to Δ_c^+. It follows from Lemma 2.7 that σ^m takes Δ^+ to Δ^- and so must equal w_0. The proof of (ii) is similar. Assertion (iii) follows from (i) and (ii) because w_0 is an involution. Finally it is easy to check from the formulae in [24, 2.2] or directly that Δ_c is just two $\langle\mathbf{r}\rangle$ orbits each of cardinality c. Then by Lemma 2.7 the cardinality of any $\langle\sigma\rangle$ orbit in Δ must be at least c. Yet it is at most c by (iii). Finally one may easily check that the two $\langle\mathbf{r}\rangle$ orbits in Δ_c are generated by a and $-b$. Then the last part follows from Remark 2.7. $\square$

Remarks. The result in (i) may also be found in Bourbaki [5] with which our proof has some common elements. It follows from (iii) that ψ is an isomorphism of $\widehat{C}$ onto C. Henceforth we write $\widehat{C}$ simply as C.

2.9. We may now give a proof of Lemma 2.2. It is clearly enough to prove its assertion for the unique longest element of C. Written as a reduced decomposition in C and then viewed as an element of W it is a product of $m\ell$ simple reflections in case (i) above and a product of either $m\ell + \operatorname{card}\pi_a$ or $m\ell + \operatorname{card}\pi_b$ simple reflections in case (ii) above. Yet by the conclusion of Lemma 2.8 it equals w_0 whose reduced length is $\operatorname{card}\Delta^+$. It follows in both cases that $c\ell \leqslant$

card Δ with equality if and only if these expressions for w_0 are reduced. On the other hand by (iii) and (iv) of Lemma 2.8, it follows that card $\Delta \leqslant c\ell$ with equality only if c has exactly ℓ orbits in Δ. Combining these inequalities proves Lemma 2.2 and moreover we also obtain that card $\pi_a = \text{card}\, \pi_b$ in case (ii).

We may thus conclude that every $\langle\sigma\rangle$ orbit meets $\pi_a \sqcup -\pi_b$ at exactly one point. Of course a σ orbit is just a $\langle\sigma^{-1}\rangle$ orbit, whilst replacing σ by σ^{-1} corresponds to interchanging a and b. Consequently every $\langle\sigma\rangle$ orbit also meets $\pi_b \sqcup -\pi_a$ at exactly one point, a fact one may also prove in a similar manner to the first statement.

The last result above is a fairly easy consequence of the results of Coleman [6] and Kostant [26, Thm. 8.1], for our particular choice of Coxeter element. It implies that there exists a bijection of $\pi_a \sqcup -\pi_b$ onto $\pi_b \sqcup -\pi_a$ implemented by sweeping out the σ orbits. Lemma 2.8 further shows that this bijection is exactly that implemented by the action of w_0. This does not quite follow from Coleman/Kostant alone.

2.10. The proofs in the above section do have some case by case nature. However since w_0 only exists for finite Weyl groups it can in principle be turned around to give a classification of Dynkin diagrams for which the latter holds, though we have to admit that we did not attempt this. Again the construction of 2.6 can easily be adapted to the case when the vertices of the Dynkin diagram is broken into several totally disconnected sets $\{\pi_s\}_{s\in S}$, for some index set S. This gives rise to (the image of) a Coxeter group C on card S generators. Let $-x_{s,t}$ denote the entry in the Cartan matrix corresponding to an ordered pair of distinct elements of S. Then Eq. $(*)$ of 2.6 is modified in the following way. First $-x$ on the left-hand side is replaced by $-x_{s,t}$, secondly i is taken to be in π_t, thirdly j in the sum is restricted to π_s. Then a solution to this equation gives a homomorphism of C into W. This generally has a kernel since C will be infinite whilst W is finite. However in the case of the two remaining finite Coxeter groups, customarily denoted as H_3, H_4, Hoyt [17] verified that they may be respectively embedded into the Weyl groups in types D_6, E_8 respectively. The calculation is apparently much easier than for example some earlier constructions using actions (see for example, [30]).

2.11. Let κ denote $-w_0$ viewed as a Dynkin diagram automorphism. Since the decomposition of π given in 2.2 is unique up to permutation, it follows that either κ sends π_a (resp. π_b) to itself or it interchanges them. The latter happens just in type A_{2m}; but we shall be coy for the moment and pretend that we do not know this. We shall call the first case (1) and the second case (2). It is clear from Lemma 2.8 that case (1) is exactly when the Coxeter number is even because this is just when $\psi(w_0)$ is a rotation by π, that is equals -1.

Suppose we are in case (1). Then w_0 sends π_a to $-\pi_a$ and belongs to $\langle\sigma\rangle$.

Suppose we are in case (2). Then w_0 sends π_a to $-\pi_b$ and belongs to C.

In both cases we set $m = [c/2]$.

2.12. Notice our convention of choosing the Cartan matrix $\mathscr{A}$ to be symmetric implies that all roots Δ_c have the same length. However the image of π_a (resp. π_b) under ψ form positive multiples of a (resp. b) of differing lengths. Let ϖ_a (resp. ϖ_b) be the fundamental weight corresponding to a (resp. b). Observe that both are proportional to elements of Δ_c if and only if c is even.

Let $\{\varpi_i\colon i \in I\}$ denote the set of fundamental weights corresponding to π. Recall the $g_i\colon i \in I$ defined in 2.6.

Lemma. *For all $i \in I$ one has*

$$\psi(\varpi_i) = \begin{cases} g_i \varpi_a\colon & i \in I_a, \\ g_i \varpi_b\colon & i \in I_b. \end{cases}$$

Proof. Suppose $i \in I_a$. Then $\sigma_b \varpi_i = \varpi_i$. Then by Lemma 2.7, $\mathbf{r}_b \psi(\varpi_i) = \psi(\varpi_i)$ and so $b^\vee(\psi(\varpi_i)) = 0$, that is $\psi(\varpi_i)$ is proportional to ϖ_a. Let us write $\psi(\varpi_i) = g_i' \varpi_a$. Then $\mathbf{r}_a \psi(\varpi_i) = g_i'(\varpi_a - a)$. Yet by Lemma 2.7 the left-hand side equals $\psi(\sigma_a \varpi_i) = \psi(\varpi - \alpha_i) = g_i' \varpi_a - g_i a$. Equating these expressions gives the assertion when $i \in I_a$. The second case is similar. $\square$

2.13. Definition of $w_\subset$: c even. Assume that we are in case (1), so then $c = 2m$. If m is even we write $m = 2n$. Set $w_\subset = \sigma^n = w_\supset$. Then $w_\subset w_\supset = w_0$. In this case $\psi(w_\subset) = \mathbf{r}^n$ is a counterclockwise rotation by $\pi/2$, through the first paragraph of 2.8.

If m is odd we write $m = 2n + 1$. Set $w_\subset = \sigma^n \sigma_a$, $w_\supset = \sigma_b \sigma^n$. Then $w_\subset w_\supset = w_0$. Moreover in this case both $w_\subset$ and $w_\supset$ are involutions which must furthermore commute since w_0 is an involution. This gives $w_\subset = w_\supset w_0$. Now $\mathbf{r}^n$ is a counterclockwise rotation by $2n\pi/2m = \pi/2 - \pi/2m$. On the other hand the angle between a and b is $\pi - \pi/2m$. Note that $\mathbf{r}^m$ is always a rotation by π.

The following result is an easy consequence of the above two assertions.

Lemma *($c = 2m$).*

(i) *If m is even (resp. odd), then $\psi(w_\subset)a$ is a positive (resp. negative) multiple of ϖ_b (resp. ϖ_a) whilst $\psi(w_\subset)b$ is a negative (resp. positive) multiple of ϖ_a (resp. ϖ_b).*

(ii) *If m is either even or odd, $\mathbf{r}^t$: $t = 0, 1, 2, \dots, m$ applied to simple root is a multiple of a fundamental weight if and only if $t = [\frac{m}{2}] = n$.*

2.14. Let β_* (resp. β_*') denote the unique highest (resp. highest short) root of Δ relative to π with the convention that $\beta_* = \beta_*'$, if π is simply-laced. Uniqueness implies that they are κ invariant. Moreover β_* (resp. β_*') is either proportional to a fundamental weight corresponding to say $\alpha_0 \in \pi$ (resp. $\alpha_0' \in \pi$) or in type A a sum of fundamental weights ϖ, ϖ' interchanged by κ, so in the *same* subset of I (that is I_a or I_b) exactly in case (1).

Assume in the remainder of this section that we are in case (1), so then $c = 2m$. Then by Lemma 2.12 above $\psi(\beta_*)$, $\psi(\beta_*')$ are proportional to fundamental weights relative to π_c. Again since the Cartan scalar product $(\,,\,)$ is strictly positive on any pair of fundamental weights we have $(\beta_*, \beta_*') > 0$.

Recall the last part of 2.11 and let α_* (resp. α_*') denote the unique simple root in the $\langle\sigma\rangle$ orbit of β_* (resp. β_*'). We shall now determine α_*.

Suppose m is even and write $m = 2n$ as before. Recall that $\mathbf{r}$ is a counterclockwise rotation by $\pi/2n$ and that $w_\subset = \mathbf{r}^n$. Now the image of β_* is a fundamental weight and the image of α_* is a simple root so by Lemma 2.13(ii), the only way they can be in the same $\langle\sigma\rangle$ orbit is for $w_\subset(\alpha_*) = \beta_*$, *up to as sign*. Similarly $w_\subset(\alpha_*') = \beta_*'$, *up to a sign*.

Suppose m is odd and write $m = 2n + 1$ as before. Now $\mathbf{r}$ is a counterclockwise rotation by $\pi/2m$. Thus by Lemma 2.13(ii), the only way that β_* and α_* can be in the same $\langle\sigma\rangle$ orbit is for $\mathbf{r}^n\psi(\beta_*)$ to be $\psi(\alpha_*)$, up to a sign. On the other hand $\psi(w_\subset)a = -\mathbf{r}^n a$, whilst $\psi(w_\subset)a =$

$\mathbf{r}_b\mathbf{r}^n\psi(w_0)a = -\mathbf{r}_a\mathbf{r}^{n+1}b = -\mathbf{r}^{n+1}b$, since the latter is proportional to ϖ_a. We conclude that $\beta_* = w_\subset(\alpha_*)$, *up to a sign*. Similarly $w_\subset(\alpha'_*) = \beta'_*$, *up to a sign*.

When π is not simply-laced this gives an elegant manner to determine α_*. Indeed by the above we must have $(\alpha_*, \alpha'_*) \neq 0$ and so $(\alpha_*, \alpha'_*) < 0$, since they are simple roots (one can also follow the sign changes above). Thus α_* is the unique long simple root not orthogonal to a short simple root. Again since $\kappa(w_\subset) = w_\subset$, it follows from the above that $\kappa(\alpha_*) = \alpha_*$. Consequently α_* is the central simple root in type A_{2t+1}.

Notice that by Lemma 2.13 that (outside A_{2t+1}) the roots α_0 and α_* belong to the same subset of π exactly when m is odd.

It remains to compute α_* in types D and E. Choose $t \in I$ such that $\alpha_* = \alpha_t$. We claim that α_t is the unique trivalent simple root on which we recall that by Lemma 2.6, the function $i \rightarrowtail g_i$ takes its maximal value.

Our claim then obtains from the following consideration. Since β_* is the highest root, since the g_i defined in 2.6 are positive and since $\psi(\beta_*)$ is proportional to a fundamental weight (of π_c) it must have length greater than any other image of a root of Δ^+ proportional to that fundamental weight. Thus g_i must be maximal in the subset of I (that is I_a or I_b) to which α_* belongs. It remains to show that this subset always contains the trivalent root.

Since we are outside type A, there is just one simple root, namely α_0 not orthogonal to β_*. In Bourbaki [5, Planches I-X], this is listed as the root attached to the extra root of the extended Dynkin diagram. From this one easily checks in types D and E, that α_t and α_0 lie in the same subset of π exactly when m is odd. We claim that this is also true of the pair α_0, α_* (so then α_* and α_t will invariably be in the same subset of π). Now β_* is proportional to ϖ_{α_0} as well as being proportional to $w_\subset\alpha_*$. Thus applying ψ, our claim follows from Lemmas 2.12 and 2.13(i).

Remark. One may further check that in all types (in case (1)) the function $i \rightarrowtail g_i$ takes its maximal value on α_*.

2.15. Set $\Delta_* := \{\gamma \in \Delta \mid (\beta_*, \gamma) = 0\}$, which is a root subsystem of Δ, though not necessarily indecomposable. The Kostant cascade $\mathscr{K}_\Delta$, or simply $\mathscr{K}$, for Δ is the subset of Δ^+ defined inductively by $\mathscr{K}_\Delta = \{\beta_*\} \cup \mathscr{K}_{\Delta_*}$, where $\mathscr{K}_{\Delta_*}$ is defined to be the union of the Kostant cascades of its indecomposable components. Notice that the Kostant cascade consists of positive roots which are strongly orthogonal (that is no sum or difference is a non-zero root). It is equipped with a natural partial order defined inductively by taking β_* to be the unique largest element. There is an element h_* of $\mathfrak{h}$ such that $h_*(\beta) = 1$, for all $\beta \in \mathscr{K}$. To ensure uniqueness we impose that $h_*\kappa = h_*$ in case (1). Then it is easy to compute h_* inductively. This procedure is less appropriate in case (2) since otherwise h_* will not then take integer values on all the roots. In this case which is type A_{2m}, we choose $h_* = \varpi^\vee_{m+1}$. We shall say that a sum of (fundamental) coweights $\varpi^\vee_i$: $i \in I$, is alternating if every fundamental coweight appears with coefficient in the set $\{1, 0, -1\}$, and in addition the sign is constant on I_a and takes the opposite sign on I_b. Surprisingly h_* is an alternating sum of coweights.

Fix a positive integer $t \leqslant [c/2] = m$ and let $\sigma^{(t)}$ denote the unique reduced element of C of length t admitting a factor of σ_a on the right.

If we are in case (1) we shall take α_* to belong to π_a. In case (2), which is type A_{2m}, to be compatible with our previous choice of h_*, we choose $I_a = \{2, 4, \dots, 2m\}$. However note that in this case $\alpha_* \in I_b$. Let h_0 denote the unique alternating sum of coweights in which the coefficient of $\varpi^\vee_i$ is 1 (resp. -1) if $i \in I_b$ (resp. $i \in I_a$).

The above choices are the second reason why a and b are not interchangeable (cf. 2.5).

It is a convenient and easy fact that there exists an integer $s \geqslant 1$ such that $\sigma^{(t)}h_0$ is an alternating sum of coweights for all $t \leqslant 2s+1$ and that $\sigma^{(s+1)}h_0 = \sigma^{(s)}h_0$ is a fundamental coweight so in particular $\sigma^{(2s+1)}h_0 = h_0$. (On the other hand for $t > 2s+1$ it is generally false that $\sigma^{(t)}h_0$ is an alternating sum of coweights. Again one must not take α_* to belong to π_b, in case (1).)

A remarkable fact is the assertion of the following quite general

Lemma. $\sigma^{(m)}h_0 = h_*$.

Proof. The proof is a little case by case. First we assume that we are not in types A or C. Then we further assume that $m = c/2$ is odd. This means that we are in types B_{2n+1}, D_{2n}, E_7, E_8, G_2. Then surprisingly enough $\sigma^{(m)}h_0 = h_0 = h_*$, which is an easy case by case computation. Now assume instead that m is even, say $m = 2n$. This means that we are in types B_{2n}, D_{2n+1}, E_6, F_4 but exclude the last two cases. Then surprisingly $\sigma^{(2n+1)}h_0 = h_0$, whilst $\sigma_b h_0 = h_*$, which proves the assertion in these cases too. Now assume that we are in cases E_6, F_4, for which we remark that $m = 6$. Then $\sigma^{(9)}h_0 = h_0$, yet still $\sigma^{(6)}h_0 = h_*$!

Adopt the Bourbaki [5] enumeration. Assume now we are in type C_2. Then $\sigma^{(2)}h_0 = \sigma^{(1)}h_0 = \varpi_2^\vee = h_*$, whilst in type C_n for $n \geqslant 3$ one checks by induction that $\sigma^{(n+1)}h_0 = \sigma^{(n)}h_0 = \varpi_n = h_*$.

Assume that we are in type A_{2m-1}. Then one checks that $\sigma^{(m)}h_0 = \sigma^{(m-1)}h_0 = \varpi_m = h_*$, $\forall m$ even, whilst $\sigma^{(m+1)}h_0 = \sigma^{(m)}h_0 = \varpi_m^\vee = h_*$, $\forall m$ odd.

Assume we are in type A_{2m}. Then $\sigma^{(m+1)}h_0 = \sigma^{(m)}h_0 = \varpi_{m+1}^\vee = h_*$, $\forall m$. $\square$

2.16. Assume that we are in case (1) so that $c = 2m$ is even. Notice that $\sigma^{(m)}$ is the inverse of $w_\subset$. By construction $h_*(\beta_*) = 1$, whilst $h_0(\alpha_*) = -1$. On the other hand $\alpha_*, \beta_*, -\beta_*$ are all in the same $\langle\sigma\rangle$ orbit. Thus in particular $w_\subset \alpha_* = \beta_*$, *up to a sign.*

Corollary. *Assume c even. Then*

$$h_*(w_\subset \alpha_i) = \begin{cases} (-1)^m: & i \in I_a, \\ (-1)^{m+1}: & i \in I_b. \end{cases}$$

Proof. By 2.15 we have $h_0 = w_\subset h_*$. If m is odd, $w_\subset$ is an involution. Otherwise $w_\subset^{-1} = w_\subset w_0$. Recall that π_a and π_b are stable under $\kappa = -w_0$. Then the assertion follows from the definition of h_0. $\square$

Remark. Observe that as a consequence $|h_*(w_\subset \alpha)| = 1$, for all $\alpha \in \pi$. Now it is easy to see that h_* takes integer values on π, so via 3.4 it would have been enough to prove that $|h_*(w_\subset \alpha)| > 0$, for all $\alpha \in \pi$. This is rather natural because the images of $w_\subset \alpha$: $\alpha \in \pi$ under ψ are proportional to fundamental weights with respect to π_c, whilst $h_*(\varpi_i) > 0$, $\forall i \in I$. However partly because ψ is only $\mathbb{Z}$ linear we were unable to make to use of this to prove the required assertion.

2.17. Definition of $w_\subset$: c odd. Assume that we are in case (2), so then $c = 2m+1$ and π is of type A_{2m}. In this case we set $w_\subset := \sigma^{(m)}$, the latter having been defined in 2.15. Then $w_\subset^{-1}\beta_* = \kappa^m(\alpha_{m+1})$. From 2.15 we obtain

Corollary.

$$h_*(w_\subset \alpha_i) = \begin{cases} -1: & i \in I_a, \\ 1: & i \in I_b. \end{cases}$$

2.18. Suppose $c = 2m + 1$.

Lemma. *Suppose either m is odd and $i \in I_a$, or m is even and $i \in I_b$. Then*

$$w_0 w_\subset^{-1} \varpi_i = w_\subset^{-1} \varpi_i. \tag{$*$}$$

Proof. If m is odd, then $w_\subset$ is an involution and $w_\subset \sigma_b w_\subset = w_0$, by Lemma 2.8 and the definition of $w_\subset$. Then the first assertion results since $\sigma_b \varpi_i = \varpi_i$, $\forall i \in I_a$. If m is even $w_\subset^{-1} \sigma_\alpha w_\subset = w_0$. This similarly gives the second assertion. $\square$

2.19. A further advantage of using the Coxeter subgroup C of W is that it allows one to give a canonical description of the element $w_\subset$. First let us collect some easy general facts.

Recall that κ leaves π_a, π_b stable if c is even and induces a bijection of π_a onto π_b is c is odd.

Furthermore for c even (resp. odd) $C\pi_c$ is two (resp. one) C orbit(s).

Finally take $d \in \Delta_c$. Then $Stab_C d$ is generated by an involution if c is even and is trivial if c is odd.

Lemma.

(i) *If c is even, every C orbit meets π (resp. $-\pi$) at exactly one point.*
(ii) *If c is odd, every C orbit meets π/κ (resp. $-\pi/\kappa$) at exactly one point.*

Proof. By 2.9, every $\langle\sigma\rangle$ orbit meets $\pi_a \cup -\pi_b$ at exactly one point. Then the assertions follows from the above general facts. $\square$

2.20. Call an element of C even (resp. odd) if its reduced length as an element of C is even (resp. odd). Observe that in all cases the parity of $w_\subset$ is the same as the parity of m and that $w_\subset^{-1}\beta_* \in (-1)^m \pi$ if c is even, whilst $w_\subset^{-1}\beta_* \in \pi$, if c is odd.

Suppose c even. We claim that $w_\subset$ is the unique element of C with the above two properties. This follows from 2.19(i) and because $Stab_C \psi(\beta_*)$ is generated by a simple reflection.

Suppose c is odd. Then in the above there are two such elements, one being obtained from the other via the unique element of C taking a to b. Through κ it is immaterial in case (2) which one chooses.

Lemma. *Suppose c is odd. Then there is no root orthogonal to every root in π_a, or orthogonal to every root of π_b. Conversely if c is even, there is a root orthogonal to either every root of π_a or every root of π_b.*

Proof. Take c odd. Suppose for example that $\gamma \in \Delta$, is orthogonal to every root in π_a. Then $\psi(\gamma) = \psi(\sigma_a \gamma) = \mathbf{r}_a \psi(\gamma)$, by Lemma 2.7. This contradicts the third general fact in 2.19.

For c even the highest root satisfies the conclusion of the lemma, noting that (cf. 2.1) we exclude type A_1. $\square$

2.21. Set $\Pi = w_{\subset}\pi$ and $\Pi_c = w_{\subset}\pi_c$.

Lemma. *Suppose π not of type A_2, then Π meets neither π nor $-\pi$.*

Proof. Via 2.7 it is enough to prove the corresponding assertions for the pair Π_c, π_c. For c even, this results from Lemma 2.13. For c odd suppose that $wa = a, b, -a, -b$ respectively, for some $w \in C$. Then from the first and third of the general facts in 2.19, we can compute w to be $Id, w_0\sigma_a, \sigma_a, w_0$. Comparison with $w_{\subset}$ given in 2.17 shows that this is only possible in type A_2, with $w_{\subset} = \sigma_a$. $\square$

3. A formula for the degrees of $Y(\mathfrak{n})$ generators

3.1. Let us recall the description of the generators of the polynomial algebra $Y(\mathfrak{n})$. From [18, 3.6,4.2,4.5] it follows that the weight vectors of $Y(\mathfrak{n})$ which are irreducible as polynomials form a set of free generators. Let a be such a generator and let wt a denote its weight. By [18, 2.8, 4.12] it follows that there exist $i \in I$ and $\varepsilon_i \in \{\frac{1}{2}, 1\}$ such that

$$\mathrm{wt}\, a_i = \varepsilon_i(\varpi_i + \varpi_{\kappa(i)}). \tag{$*$}$$

The elements a_i and $a_{\kappa(i)}$ are the same. In particular these generators of $Y(\mathfrak{n})$ are in bijection with the κ orbits in π. It can be determined by [18, 4.12] exactly when $\varepsilon_i = 1$. For example it is clear that this must hold if $i \neq \kappa(i)$. Again it is true for all $i \in I$ in types A and C. For more details, see 4.4 and 4.9.

Kostant had explained to me (at Luminy in 2000) that he had constructed an element of $Y(\mathfrak{n})$ of weight $\varpi_i + \varpi_{\kappa(i)}$ through the Hopf dual of $U(\mathfrak{g})$. This element is either a generator or the square of a generator. This construction of Kostant was exploited in [13, Sect. 6] and will be used here also. The fact that ε_i may be equal to $\frac{1}{2}$ is the source of several difficulties in the description of invariants in the parabolic (and biparabolic) case. It is the reason why the upper and lower bounds in [13] do not always coincide. It is rather satisfying that these difficulties do not affect our metaslice.

Each a_i is a homogeneous element of $Y(\mathfrak{n})$ and we let deg a_i denote its degree. These were tabulated in [18, Tables I, II] with some errors and omissions which were corrected in [13, Table]. Set $\Lambda_0 := \{\mathrm{wt}\, a_i\colon\ i \in \pi/\kappa\}$.

Fix a simple finite dimensional $\mathfrak{g}$ module $V(\mu)$ of highest weight μ. Given $v \in V(\mu)$, $\xi \in V(\mu)^*$, let $b_{\xi,v}$ denote matrix coefficient in the Hopf dual $U(\mathfrak{g})^\star$ of $U(\mathfrak{g})$ sending $x \in U(\mathfrak{g})$ to $\xi(xv)$. When v is the highest weight vector (of weight μ of $V(\mu)$) and ξ the lowest weight vector (of weight $w_0\mu$) we write $b_{\xi,v}$ simply as $b_{\mu - w_0\mu}$, where the subscript now designates its weight as an $\mathfrak{h}$ module for the adjoint action, or simply, its "diagonal weight".

Now let $\mathscr{F}$ denote the filtration on $U(\mathfrak{g})^\star$ induced by the canonical filtration on $U(\mathfrak{g})$. Then $gr_{\mathscr{F}}U(\mathfrak{g})^\star$ identifies with the graded dual of $S(\mathfrak{g})$, which is just $S(\mathfrak{g}^*)$ and hence through the Killing form with $S(\mathfrak{g})$. Moreover as noted in [14, 3.5] a result of Kostant asserts that $a_{\mu - w_0\mu} := gr_{\mathscr{F}} b_{\mu - w_0\mu}$ lies in $Y(\mathfrak{n})$, after the above identifications. Again $\mathscr{F}$ is compatible with adjoint action and so this element has weight $\mu - w_0\mu$. (On the other hand if we take a linear combination b of such elements, then $gr_{\mathscr{F}} b$ lies in $Sy(\mathfrak{b})$, see [14, 3.6].)

This construction gives all the above generators for which $\varepsilon_i = 1$. Since $Y(\mathfrak{n})$ is multiplicity free, the remaining generators must be square roots of those having twice the weight. The precise conditions under which such square roots exist was established in [18, 4,12], independent of the above, and in [19, 7.5.5] using the above, some easy additional information on the weights of the quotient field of $Y(\mathfrak{n})$ and the fact that $Y(\mathfrak{n})$ is a unique factorization domain. One finds that $a_{\mu-w_0\mu}$ has a square root in $Y(\mathfrak{n})$ exactly when $\frac{1}{2}(\mu-w_0\mu)$ is a $\mathbb{Z}$ linear combination of elements from a Kostant cascade. For this it is necessary but not sufficient that $w_0\mu=-\mu$.

In the above deg b: $b\in U(\mathfrak{g})^\star$ is defined to be the least integer n such that $b(\mathscr{F}^nU(\mathfrak{g}))\neq 0$. If $b\in U(\mathfrak{g})^\star$ has degree n, then by definition of degree $b(y^m)\neq 0$: $y\in\mathfrak{g}$ implies $m\geqslant n$ and moreover $(gr_{\mathscr{F}}b)(y)\neq 0$ exactly when $m=n$. Notice that with respect to the standard gradation on $S(\mathfrak{g})$, the element $gr_{\mathscr{F}}b$ is homogeneous of degree deg b.

3.2. In case (1) we define

$$s(i)=\begin{cases}-1: & i\in I_a,\\ 1: & i\in I_b.\end{cases}$$

In case (2) we define

$$s(i)=\begin{cases}2: & \text{either } m \text{ is even and } i\in I_a, \text{ or } m \text{ is odd and } i\in I_b,\\ 0: & \text{either } m \text{ is odd and } i\in I_a, \text{ or } m \text{ is even and } i\in I_b.\end{cases}$$

Define $h\in\mathfrak{h}$ by

$$h(\varpi_i)=s(i)\deg a_i^{1/\varepsilon_i},\quad \forall i\in I.$$

In case (1), one has $s(\kappa(i))=s(i)$, $\forall i\in I$ and so $h\kappa=h$, which may also be written as $\kappa(h)=h$, for the induced action of κ on $\mathfrak{h}$. Moreover $h(\mathrm{wt}\,a_i)=2s(i)\deg a_i$, $\forall i\in I$. Furthermore these two properties define h.

In case (2), κ induces a bijection of I_a onto I_b and so $\varepsilon_i=1$, $\forall i\in I$. In particular $h(\mathrm{wt}\,a_i)=2\deg a_i$, $\forall i\in I$. This and the further property $h(\varpi_i)=0$, if either m is odd and $i\in I_a$, or m is even and $i\in I_b$, define h.

Recall the element $w_\subset\in W$ defined in 2.13, 2.17. The aim of the present section is to prove the quite astonishing

Theorem. *One has $h(w_\subset\alpha_i)=2$, $\forall i\in I$ and $w_\subset$ is the unique element of W with this property.*

3.3. One consequence of Theorem 3.2 is that we can compute the degrees of the generators from their weights. The proof results mainly from the proposition below which is the first part of what is needed to recover a metaslice. This does not determine the mysterious ε_i factors. However this will be achieved in the second part (Theorem 5.12).

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. Then the key consequence of the above theorem is that $h':=w_\subset h$ is just the semisimple element of the standard principal s-triple (x',h',y'), which we define by the relations $[h',x']=-2x'$, $[h',y']=2y'$, $[x',y']=-h'$ and the condition that (x',h',y') are regular elements and in $(\mathfrak{n},\mathfrak{h},\mathfrak{n}^-)$, respectively. In particular we can write

$$y'=\sum_{i\in I}x_{\alpha_i},$$

and then y is given by $y=w_\subset y'$.

This convention of using y' as having a positive ad h eigenvector avoids some superfluous sign changes. Of course we could also replace y' by x'; but this might cause some confusion with the generic symbol for the Chevalley basis.

In some circumstances (for example in type A_{2m}) the element y restricted to $\mathfrak{b}$ becomes a regular element of $\mathfrak{b}^*$ and by further truncating h in an obvious way we obtain an adapted pair (h, y) in the sense of [22, 2.7], up to a scale factor of -2.

An interesting question is whether any such adapted pair can be so obtained and in particular expressed through an element of the Weyl group as above.

There is no fundamental difficulty in verifying Theorem 3.2 case by case. However an understanding of the form of $w_\subset$ permits one to give a proof which is mainly case by case free. Roughly speaking $w_\subset$ should be viewed as the "square root" of w_0 and that it takes h' to an element which is midway between h' and $-h'$, making roughly speaking the sum of the heights of the elements of $w\pi$ to be maximal.

In order to prove Theorem 3.2 it is nevertheless necessary to consider the cases (1) and (2) separately.

Assume we are in case (1), so then $c = 2m$ and $w_0 = \sigma^m$, by 2.8(i).

Recall the notation of 2.12. Assume that $m = 2n$ and recall that $w_\subset = \sigma^n = w_\supset$. Observe that $w_\subset \sigma_b w_\subset^{-1} = w_0 \sigma_b$ which implies that

$$w_\subset \sigma_b w_\subset^{-1} \varpi_i = w_0 \varpi_i, \quad \forall i \in I_a. \tag{1}$$

Similarly

$$w_\subset \sigma_a w_\subset^{-1} \varpi_i = w_0 \varpi_i, \quad \forall i \in I_b. \tag{2}$$

Now suppose that $m = 2n + 1$ and recall that $w_\subset = \sigma^n \sigma_a$, $w_\supset = \sigma_b \sigma^n$ and $w_\subset w_\supset = w_0$. Observe that $w_\subset \sigma_b w_\subset^{-1} = w_0 \sigma_a$, which implies that

$$w_\subset \sigma_b w_\subset^{-1} \varpi_i = w_0 \varpi_i, \quad \forall i \in I_b. \tag{3}$$

Similarly

$$w_\subset \sigma_a w_\subset^{-1} \varpi_i = w_0 \varpi_i, \quad \forall i \in I_a. \tag{4}$$

3.4. Set $\Pi_a = w_\subset \pi_a$, $\Pi_b = w_\subset \pi_b$, so then Π is their union, that is $\Pi = w_\subset \pi$. Let Σ_a (resp. Σ_b) be the product of the reflections Π_a (resp. Π_b). Set

$$y_a = \sum_{\gamma \in \Pi_a} x_\gamma, \qquad y_b = \sum_{\gamma \in \Pi_b} x_\gamma, \qquad y = \sum y_a + y_b. \tag{$*$}$$

Recall [18, 4.4], that every weight subspace $Y(\mathfrak{n})_\mu$: $\mu \in \mathfrak{h}^*$ has dimension $\leqslant 1$ and let Λ denote the subset of $\mathfrak{h}^*$ for which equality holds. One calls Λ the set of weights of $Y(\mathfrak{n})$. Of course $\Lambda \subset P^+$ and is the free abelian semigroup generated by Λ_0. Similarly one may define the set of weights $Y(\mathfrak{n}^-)$. It is clear that this equals $-\Lambda$. Given $\mu \in \Lambda$ recall that there is a unique up to scalars element a_μ (resp. $a_{-\mu}$) of $Y(\mathfrak{n})$ (resp. $Y(\mathfrak{n}^-)$) of weight μ (resp. $-\mu$).

Proposition. *Given $\mu \in \Lambda_0$, then either $a_\mu(y) \neq 0$ or $a_{-\mu}(y) \neq 0$.*

Proof. First assume we are in case (1) and recall the notation of 3.3.

There are four subcases to consider depending on whether m is even or odd and whether i belongs to I_a or to I_b. We shall treat one subcase in detail and explain the (very minor) changes needed for the other three subcases. Actually it is possible to treat all subcases simultaneously as indicated in the analogous calculation in Section 4. However we believe that for the first time around this is less comprehensible.

Assume that $m = 2n$. Then $\Pi_a \subset \Delta^+$ and $\Pi_b \subset \Delta^-$. The first (resp. second) assertion results from the fact that $n + 1 \leqslant 2n$ and so $\sigma^n \sigma_a$ (resp. σ^n) is a reduced expression in C and hence by Lemma 2.2 a reduced expression in W. Since $a_\mu \in Y(\mathfrak{n})$ and $a_{-\mu} \in Y(\mathfrak{n}^-)$, it follows that $a_\mu(y) = a_\mu(y_b)$ and $a_{-\mu}(y) = a_{-\mu}(y_a)$.

Replacing μ by 2μ if necessary we can assume that $\mu = \varpi_i - w_0\varpi_i$ for some $i \in I$. Take $i \in I_a$. Then by (1)

$$\Sigma_b \varpi_i = w_0 \varpi_i.$$

This implies (the somewhat remarkable fact) that $\varpi_i - w_0\varpi_i$ is a sum with non-positive integer coefficients of the roots from Π_b which we recall are pairwise orthogonal. Precisely (for m even)

$$\varpi_i - w_0\varpi_i = \sum_{\gamma \in \Pi_b} \gamma^\vee(\varpi_i)\gamma, \quad \forall i \in I_a. \tag{5}$$

Moreover the coefficient $m_i(\gamma)$ of $-\gamma \in -\Pi_b \subset \Delta^+$ is just $-\gamma^\vee(\varpi_i)$, which is a non-negative integer. Let k_i denote the sum of these coefficients. By 2.16 since m is even we obtain $k_i = h_*(\varpi_i - w_0\varpi_i)$.

We claim that the expansion of $y_b^{k_i} v_{\varpi_i}$ as weight vectors contains a non-zero multiple of the lowest weight vector $v_{w_0\varpi_i}$ of $V(\varpi_i)$. Indeed recall that the root vectors which sum to y_b commute so then by $(*)$ and their linear independence there is just one monomial in $y_b^{k_i}$, that is

$$\prod_{\gamma \in \Pi_b} x_\gamma^{m_i(\gamma)},$$

which applied to v_{ϖ_i} can give a non-zero multiple of $v_{w_0\varpi_i}$. It does give a non-zero multiple by standard $\mathfrak{sl}(2)$ theory, applied to each $\mathfrak{sl}(2)$ triple defined by $\gamma \in \Pi_b$. Indeed since v_{ϖ_i}: $i \in I_a$ is a highest weight vector, successive applications of $x_\gamma^{\gamma^\vee(\varpi_i)}$: $\gamma \in \Pi_b$, taken in any order, carry extremal vectors to extremal vectors in $V(\varpi_i)$. Thus our claim follows.

Now let $b_{-w_0\varpi_i,\varpi_i}$ (or simply, $b_{(i)}$) denote the matrix coefficient in the Hopf dual $U(\mathfrak{g})^\star$ of $U(\mathfrak{g})$ defined by

$$b_{-w_0\varpi_i,\varpi_i}(a) = \xi_{-w_0\varpi_i}(av_{\varpi_i}),$$

where ξ_ν denotes a vector in $V(\mu)^*$ of weight ν. Then the above claim translates to give

$$b_{(i)}\bigl(y_b^k\bigr) \neq 0. \tag{6}$$

Now recall the (decreasing) filtration $\mathscr{F}$ on $U(\mathfrak{g})^\star$ induced by the canonical filtration on $U(\mathfrak{g})$ and that we may identify $gr_{\mathscr{F}} b_{-w_0\varpi_i,\varpi_i}$ with an element $a_{\varpi_i - w_0\varpi_i}$ of $Y(\mathfrak{n})$ of weight

$\varpi_i - w_0\varpi_i$. Since $Y(\mathfrak{n})$ is multiplicity free the latter is just a non-zero multiple of $a_{\varpi_i - w_0\varpi_i} = a_i^{1/\varepsilon_i}$.

From the definition of degree given in 3.1 it follows in particular that $\deg a_{\varpi_i - w_0\varpi_i} \leqslant k_i$ and that equality implies $a_{\varpi_i - w_0\varpi_i}(y) \neq 0$.

Let us show that equality does in fact hold. Of course this is a delicate point and we do not have a proof which is as natural or easy as we would have liked.

From our description of the generators of $Y(\mathfrak{n})$, it follows that each $\mu \in \Lambda$ is a linear combination of the element of the Kostant cascade $\mathscr{K}$ with non-negative integer coefficients. Let m_μ be the sum of these coefficients. By [18, 4.12(iii)], it follows the corresponding unique up to scalars element $a_\mu \in Y(\mathfrak{n})$ of weight μ has degree equal to m_μ. By definition of h_* one has $h_*(\mu) = m_\mu$. Now take $\mu = \varpi_i - w_0\varpi_i$: $i \in I_\alpha$. Then $k_i = h_*(\varpi_i - w_0\varpi_i) = \deg a_i^{1/\varepsilon_i}$, which is what we needed to prove.

To complete the proof of the proposition in case (1) we must consider three further possibilities. First that m is even and $i \in I_b$ and then that m is odd and either $i \in I_a$ or $i \in I_b$. However it is clear from (2)–(4) above and Corollary 2.16 that the proof in these subcases is exactly the same, except for some rather irritating sign changes. Because of the latter we set

$$|\gamma| = \begin{cases} \gamma: & \gamma \in \Delta^+, \\ -\gamma: & \gamma \in \Delta^-. \end{cases} \tag{$*$}$$

(This notation will be used systematically in Section 4.) Then taking $m_i(\gamma)$: $\gamma \in \Pi_a$ to be $|\gamma|^\vee(\varpi_i)$, which is a non-negative integer, the analysis proceeds as before.

This completes the proof of the proposition in case (1). $\square$

3.5. Continue to assume that we are in case (1) so then $c = 2m$ is even. Provisionally define $h \in \mathfrak{h}$ by $h(w_\subset \alpha_i) = 2$, $\forall i \in I$. We show that h coincides with the element defined in 3.2. This will prove the first part of Theorem 3.2. Since $\kappa(w_\subset) = w_\subset$, when c is even, it follows that $\kappa(h) = h$. Suppose that m is even and take $i \in I_a$, so then $s(i) = -1$. Then by (5) above

$$h(\varpi_i - w_0\varpi_i) = \sum_{\gamma \in \Pi_b} -m_i(\gamma)h(\gamma) = -2k_i = 2s(i)\deg a_i^{1/\varepsilon_i}.$$

This result also holds in the three further possibilities which must be considered (as noted in the last part of the above proposition). Indeed irrespective of whether m is odd or even one always has $k_i = \deg a_i^{1/\varepsilon_i}$ and

$$h(\varpi_i - w_0\varpi_i) = \begin{cases} -2k_i: & i \in I_a, \\ 2k_i: & i \in I_b. \end{cases} \tag{$*$}$$

Comparison of this formula with the definition of h given in 3.2 we conclude that in case (1) the conclusion of the Theorem 3.2 holds with $w = w_\subset$ and no other since h is regular. This completes the proof of the Theorem 3.2 in case (1).

3.6. Assume we are in case (2), so then $c = 2m + 1$. Recall that $w_\subset = \sigma^{(m)}$ in this case. Then by Lemma 2.8(ii), we obtain

$$w_\subset \sigma_b w_\subset^{-1} = w_0. \tag{7}$$

Recall the definitions of 3.4; noting that now $y_a \in \mathfrak{n}^-$ and $y_b \in \mathfrak{n}$. Thus $a_{-\mu}(y) = a_{-\mu}(y_b)$.

Now the proof of Proposition 3.4 is exactly as before. In a little more detail for all $i \in I$, let k_i denote the sum of the integers $\{\gamma^\vee(\varpi_i)\colon \gamma \in \Pi_b\}$ which we note are all non-negative. Then as before through (7) it follows by $\mathfrak{sl}(2)$ theory as in 3.4 that $y_b^{k_i} v_{-\varpi_i}$ is a non-zero multiple of $v_{-w_o\varpi_i}$. Moreover $k_i = h_*(\varpi_i - w_0\varpi_i)$ by Corollary 2.17. On the other hand from the definition of h_* and the expression for $\deg a_i$ given in [18], we obtain $\deg a_i = h_*(\varpi_i - w_0\varpi_i)$ also. Hence as before $a_{-\mu}(y_b) \neq 0$. This establishes Proposition 3.4 in case 2 and hence completes its proof.

To establish Theorem 3.2 in case (2), we again provisionally define $h \in \mathfrak{h}$ by $h(w_{\subset}\alpha_i) = 2$, $\forall i \in I$. Recall that $a_i = a_{\varpi_i - w_0\varpi_i}$ and so $h(\mathrm{wt}\, a_i) = h(\varpi_i - w_0\varpi_i)$. Then by Eq. (5) of 3.4, we obtain

$$h(wta_i) = \sum_{\gamma \in \Pi_\beta} \gamma^\vee(\varpi_i)h(\gamma) = 2k_i = 2\deg a_i.$$

It remains to show that $h(\varpi_i) = 0$ if either m is odd and $i \in I_a$, or m is even and $i \in I_b$. Now when either of these hold we obtain $\kappa(w_{\subset}^{-1}\varpi_i) = -w_{\subset}^{-1}\varpi_i$ by Lemma 2.18. On the other hand by (the provisional) definition of h one has $(w_{\subset}^{-1}h)(\alpha_i) = 2$, $\forall i \in I$ by definition of h. Hence $\kappa(w_{\subset}^{-1}h) = w_{\subset}^{-1}h$. Together these imply the required assertion. This establishes Theorem 3.2 in case (2) and hence completes its proof.

4. The choice of affine subspace

4.1. We now turn to the description of the vector space V which is the second component in the description of the affine subspace $y + V$ forming the metaslice of the Borel. Just as the choice of y was rather delicate, though ultimately given in an intrinsic manner, so is the choice of V. Thus y was given by a particular W translate of the simple root system π defining the Borel, and so in particular by $\operatorname{card}\pi$ linearly independent roots. Similarly V will be given as the vector space spanned by the root vectors of a set T of $\operatorname{card}\pi$ linearly independent roots whose choice we describe below.

4.2. Recall the definition of h_* given in 2.15 and set $\Delta_1 := \{\alpha \in \Delta \mid h_*(\alpha) = 1\}$. Since h_* is conjugate to a fundamental coweight the structure of Δ_1 is rather easy to understand and in particular it has cardinality $\geqslant |\pi|$. To describe V above we define an injective map θ of π/κ into Δ_1. Let π_0 be a subset of π formed from a set of representatives of each κ orbit of cardinality 2. Then for c even, T above is defined to be $\theta(\pi) \cup \pi_0$, up to signs (!) which will be specified in 5.1. For c odd further precisions are necessary.

In types A and C the description of θ is quite straightforward. Indeed for each $\alpha \in \pi$ there is a unique minimal $\beta \in \mathscr{K}$ such that $(\alpha, \beta) \neq 0$. Moreover β, being κ invariant, is independent of the choice of α in its κ orbit and we set $\theta(\alpha) = \beta$. However this solution is deceptively simple and the wrong approach in general.

4.3. We assume until 4.12 that the Coxeter number c is even and we set $c = 2m$ as before.

Recall the decomposition of the index set I of π into two (unique up to interchange) subsets I_a, I_b. Given $\gamma \in \pi$, set

$$\tau(\gamma) = \begin{cases} a\colon & \text{either } \gamma \in \pi_a \text{ and } m \text{ is even, or } \gamma \in \pi_b \text{ and } m \text{ is odd,} \\ b\colon & \text{either } \gamma \in \pi_a \text{ and } m \text{ is odd, or } \gamma \in \pi_b \text{ and } m \text{ is even.} \end{cases}$$

The reason for this definition comes from the following analogues (1)–(4) of 3.3, namely: For m even

$$w_{\subset}\sigma_b w_{\subset}^{-1}\varpi_i = w_0(\varpi_i - \alpha_i), \quad \forall i \in I_b, \tag{1}$$

and

$$w_{\subset}\sigma_a w_{\subset}^{-1}\varpi_i = w_0(\varpi_i - \alpha_i), \quad \forall i \in I_a. \tag{2}$$

For m odd

$$w_{\subset}\sigma_b w_{\subset}^{-1}\varpi_i = w_0(\varpi_i - \alpha_i), \quad \forall i \in I_a, \tag{3}$$

and

$$w_{\subset}\sigma_a w_{\subset}^{-1}\varpi_i = w_0(\varpi_i - \alpha_i), \quad \forall i \in I_b. \tag{4}$$

Following through the (easy) computation in 3.4, we can summarize the conclusion of Eqs. (1)–(4), using the above notation, in the one relation

$$\varpi_\alpha + \varpi_{\kappa(\alpha)} = \kappa(\alpha) + \sum_{\gamma \in \Pi_{\tau(\alpha)}} \gamma^\vee(\varpi_\alpha)\gamma, \quad \forall \alpha \in \pi. \tag{5}$$

Since c is supposed even, κ leaves both π_a and π_b stable (and so in particular commutes with $w_{\subset}$). Thus applying κ to the above we may deduce that

$$\varpi_\alpha + \varpi_{\kappa(\alpha)} = \alpha + \sum_{\gamma \in \Pi_{\tau(\alpha)}} \gamma^\vee\big(\kappa(\varpi_\alpha)\big)\gamma, \quad \forall \alpha \in \pi. \tag{6}$$

This is exactly what we had obtained previously (see Eq. (5) of 3.4) with two changes. First the presence of α before the summation in right-hand side of (5) and secondly that $\tau(\alpha)$ is exactly the opposite to what it was before. Ultimately it means that we shall obtain a version of Proposition 3.4 with the monomials occurring in y combined with a linear factor from V and at the same time with μ replaced by $-\mu$. However to ensure the previous degree estimates work out we must replace α in the above by an element of Δ_1. This is the role of θ.

4.4. Let us use the convention that every root is deemed long if Δ is simply-laced and that a root is deemed short only if Δ is not simply-laced. We set $r = (\gamma,\gamma)/(\alpha,\alpha)$ for γ long and α short. It is convenient to fix $(\gamma,\gamma) = 2r$, for any long root γ. If $\delta = \delta_1 + \delta_2$ with δ_1 a sum of long roots and δ_2 a sum of short roots, then $(\delta,\delta) = (\delta_2,\delta_2) \bmod 2r$ and if $\delta_1 \neq 0$, it can equal $2r$ only if δ is also a root. (The proviso $\delta_1 \neq 0$ is only needed in type C.)

The weights $Y(\mathfrak{n})$ are of course dominant and sums of elements of $\mathscr{K}$ with non-negative integer coefficients, by virtue of the Heisenberg subalgebras associated with the Kostant cascade (see [18, Sect. 2], for example). It is a little more difficult to prove (see [18, 4.12] or [19, 7.5.5]) that conversely every weight of $Y(\mathfrak{n})$ is of this form.

The highest root β_* is necessarily long (as is well known). Indeed set $\Gamma^0 := (\Delta - (\Delta_* \cup \{\beta_*\}))^+$. Then $\frac{1}{2}(\beta_*,\beta_*) = (\beta_*,\gamma)$, for all $\gamma \in \Gamma^0$, by [18, 2.2], so we can assume that Γ^0 has only shorts roots (which incidentally is just the case in type C). Again this relation and the simplicity of $\mathfrak{g}$ implies that $\{\gamma - \gamma' \mid \gamma, \gamma' \in \Gamma^0\} = \Delta_*$. Choose $\gamma, \gamma' \in \Gamma^0$ such that $\gamma - \gamma'$ is a long root. This forces $(\gamma, \gamma') \geqslant 0$ and so $\gamma + \gamma'$ is a long root and by the first relation above is necessarily β_*.

It does not quite follow from the above that every element of $\mathscr{K}$ is long. The trouble being that $\Delta_{**,\dots,*}$ inductively defined via 2.15 may be a non-empty subset of Δ consisting of only short roots. This occurs in just types $B_\ell : \ell$ odd and G_2.

Recall 3.1 and set $\varepsilon_i = \varepsilon_\alpha$, when $\alpha_i = \alpha$.

Lemma. *Suppose that $\varepsilon_\alpha = 1/2$.*

(i) *Then ϖ_α is a sum of long roots.*
(ii) *α is a long root.*

Proof. (i) is easily verified taking account of the above remarks and because $\varepsilon_\ell = 1$ in type G_ℓ: $\ell = 2$ and in type B_ℓ: ℓ odd. Finally $\alpha^\vee(\varpi_\alpha) = 1$, which had α been a short root, Eq. (6) of 4.3 would have implied it to be divisible by r in virtue of (i). Hence (ii). $\square$

4.5. Recall 2.15, that $h_*(\pi) \in \{-1, 0, 1\}$ and is constant on κ orbits. Recall the definition of Π given in 3.4 and of $|\gamma|$: $\gamma \in \Delta$ in 3.4(∗). Observe that the conclusion of Corollary 3.4 can be written as $h_*(|\gamma|) = 1$, $\forall \gamma \in \Pi$. Take $\alpha \in \pi$. If $h_*(\alpha) = 1$, we set $\theta(\alpha) = \alpha$.

To define $\theta(\alpha)$ in general we add to α certain positive roots $|\gamma|$ with $\gamma \in \Pi_{\tau(\alpha)}$ "occurring" in the right-hand side of (6). To be precise this means that when we replace α by $\theta(\alpha)$ in (6) the coefficient of $|\gamma|$ in the new right-hand side of (6) must remain non-negative. If c is even then by Eq. (6) of 4.3 one must just check that $|\gamma|^\vee(\kappa(\varpi_\alpha)) \geqslant (\alpha, \alpha)/(\gamma, \gamma)$. This is generally immediate from our construction.

Claim (1). Suppose $h_*(\alpha) = 0$. Then there exists a unique $\gamma \in \Pi_{\tau(\alpha)}$ such that $|\gamma| + \alpha$ is a positive root. Moreover $|\gamma|^\vee(\kappa(\varpi_\alpha)) \geqslant 1$ and $(\alpha, \alpha) = (\gamma, \gamma)$.

In this case we set $\theta(\alpha) = |\gamma| + \alpha$.

Claim (2). Suppose $h_*(\alpha) = -1$. (In this case we remark that κ is the identity on α (and hence on ϖ_α) and α is a long root.) Then there are three canonically determined roots $\gamma_i \in \Pi_{\tau(\alpha)}$: $i = 1, 2, 3$ not necessarily distinct such that $\frac{1}{2}(|\gamma_1| + |\gamma_2| + |\gamma_3| + \alpha)$ is a positive root. To be precise when $r = 1$ the above elements are exactly those in $\Pi_{\tau(\alpha)}$, for which the coefficient m_i^α of α is odd. Then in particular $|\gamma_i|^\vee(\varpi_\alpha) \geqslant m_i^\alpha \geqslant 1$, $\forall i = 1, 2, 3$. If $r = 2$, then $\gamma_2 = \gamma_3$ and is the unique short root in $\Pi_{\tau(\alpha)}$, whilst γ_1 is the unique long root in $\Pi_{\tau(\alpha)}$, in which α appears with an odd coefficient, so of course $|\gamma_1|^\vee(\varpi_\alpha) \geqslant 1$. If $r = 3$ then $\gamma_1 = \gamma_2 = \gamma_3$ and is the unique short root in $\Pi_{\tau(\alpha)}$. Further for this short root γ one has $|\gamma|^\vee(\varpi_\alpha) \geqslant r$.

In this case we take the above root to be $\theta(\alpha)$. It is canonically determined by α.

In all cases $h_*(\theta(\alpha)) = 1$.

The above description of θ is quite intrinsic in that we do not have to specify to which simple algebra we are referring. However the above claims (which have to be proved!) does involve some case by case analysis which we shall try to reduce to a minimum. Indeed we shall pretend that we do not know Π explicitly though for the reader's convenience we describe Π in the Appendix.

4.6. In this section we fix a κ stable element $\alpha \in \pi$.

Lemma. *For all $\gamma \in \Pi_{\tau(\alpha)}$ one has $\gamma^\vee(\alpha) = 0$.*

Proof. Under the above hypothesis (5) of 4.3 becomes

$$\varpi_\alpha = \frac{1}{2}\left(\alpha + \sum_{\gamma \in \Pi_{\tau(\alpha)}} \gamma^\vee(\varpi_\alpha)\gamma\right). \tag{$*$}$$

Take the value of this expression on $\gamma^\vee$: $\gamma \in \Pi_{\tau(\alpha)}$. Since the elements of $\Pi_{\tau(\alpha)}$ are pairwise orthogonal we conclude that $\gamma^\vee(\alpha) = 0$, as required. □

Remark. Thus the sum in $(*)$ is an orthogonal one. More generally it is always true for c even that the elements of $\{\alpha\}, \Pi_{\tau(\alpha)}$ are linearly independent (see Remark 1 in 4.11). Thus we can speak of the number of times $|\gamma|$ appears in ϖ_α, which here is just $\frac{1}{2}|\gamma|^\vee(\varpi_{\kappa(\alpha)})$. We will use this language in verifying the last part of the claims.

4.7. We need the following rather technical

Lemma. *Let $\{\gamma', \pi''\}$ be a set of orthogonal positive roots with $\pi'' \subset \pi$, such that*

$$\delta' := \frac{1}{2}\left(\gamma' + \sum_{\alpha' \in \pi''} \alpha'\right),$$

is a sum of roots.

(i) *If $r = 1$, then Δ is of type D or E. Moreover* $\operatorname{card}\pi'' = 3$ *and δ is a root.*
(ii) *If $r > 1$, then Δ is of type B, F, G.*
(iii) *Suppose Δ of type B, G. Then π'' consists of a single root α' (long if $r = 2$, short if $r = 3$), γ' is a long root and $\delta' = \frac{1}{2}(\gamma' + \alpha')$ is a short root.*
(iv) *Suppose Δ is of type F. Then apart from the solution described in* (iii) *we may also have* $\operatorname{card}\pi'' = 2$ *with γ', δ' being short roots.*

Proof. Assume first that $r = 1$. Then $(\delta', \delta') = \frac{1}{4}(\gamma', \gamma')(1 + \operatorname{card}\pi'')$. The hypothesis that δ' is a sum of roots forces (δ', δ') to be a positive integer multiple of (γ', γ'). Consequently $\operatorname{card}\pi'' = 3 \bmod 4$.

Suppose Δ is exceptional, that is of type E. Then $\operatorname{card}\pi'' \leqslant 4$ and hence equal to 3. Thus δ' satisfies $(\delta', \delta') = (\gamma', \gamma')$ and so being a sum of roots must also be a root, via the first paragraph of 4.4.

Suppose Δ is classical. Since δ' is a sum of roots the coefficients of any $\alpha' \in \pi$ occurring in γ' must be odd (hence 1) if $\alpha' \in \pi''$ and even otherwise. Now a root cannot be orthogonal to all the simple roots in its support, so at least one simple root has coefficient 2 in γ'. In particular Δ must be of type D_ℓ. Moreover using the Bourbaki [5] notation, there exists $t \in \{1, 2, \ldots, \ell - 3\}$ such that $\pi'' = \{\alpha_t, \alpha_{\ell-1}\alpha_\ell\}$ and $\gamma' = \varepsilon_t + \varepsilon_{t+1}$. Then $\delta' = \varepsilon_t + \varepsilon_{\ell-1}$ and so is a root. This proves (i).

Assume $r > 1$.

Suppose that Δ is of type B_ℓ. Then as in type D we conclude that there exists $t \in \{1, 2, \ldots, \ell - 1\}$ such that $\pi'' = \{\alpha_t\}$ and $\gamma = \varepsilon_t + \varepsilon_{t+1}$, which is a long root. Then $\delta' = \varepsilon_t$ and is a short root. This proves that part of (iii) pertaining to type B.

Suppose that Δ is of type C_ℓ. Then as before we conclude that there exists $t \in \{1, 2, \ldots, \ell - 1\}$ such that $\pi'' = \{\alpha_t\}$ and $\gamma' = \varepsilon_t + \varepsilon_{t+1}$. However in this case we must have $\delta = \varepsilon_t$ which is not a root nor a sum of roots. This proves (ii).

Suppose that Δ is of type F_4. This forces $\operatorname{card}\pi'' \leqslant 2$. If $\pi'' = \{\alpha'\}$, then γ', α' must be long and then via the first paragraph of 4.4, δ' must be a short root. Otherwise π'' has one short root and one long root forcing γ', δ' to be short roots. This proves (iv).

Finally suppose Δ of type G_2. Then $\pi'' = \{\alpha'\}$, forcing γ' long and α', δ' to be short roots. This completes the proof of (iii). □

Remark. The exceptionality of types A, C in the above will be related to the fact that these are exactly the cases when h_* does not take the value -1 on an element of π.

4.8. In this section we shall take $i \in I$ for which $\varepsilon_i = 1/2$. By [18, 4.12] this occurs exactly when ϖ_i is a sum of roots in $\mathscr{K}$. Moreover ϖ_i is a sum of long roots by 4.4.

Set $\alpha = \alpha_i$ and $\Pi' := \Pi_{\tau(\alpha)}$.

Lemma. *Suppose $\varepsilon_\alpha = 1/2$. Then*

(i) *There exists a positive root $\theta(\alpha)$ described as in Claim* (2) *of* 4.4.
(ii) *$\theta(\alpha)$ is a long root and $(\theta(\alpha), |\gamma|) \geqslant 0$, $\forall \gamma \in \Pi_{\tau(\alpha)}$.*
(iii) *$h_*(\theta(\alpha)) \geqslant 1$.*

Proof. Under the above hypotheses, α must be κ stable and a long root, through the remarks in 3.1 and 4.4. Moreover in $(*)$ of 4.3 the expression ϖ_α must be a sum of long roots and hence so must be the right-hand side. Obviously this remains true if we subtract just long roots though it may fail if we subtract short roots. Nevertheless we subtract roots lying in Π' from ϖ_α until the roots occurring in the right-hand side of 4.6 $(*)$ all occur with coefficient $\frac{1}{2}$, which is then a sum of roots though not necessarily all long, replacing Π' by a smaller subset Π'' if necessary. Now apply $w_\subset^{-1}$ to both sides of $(*)$ of this new right-hand side and set $\pi'' := w_\subset^{-1}\Pi'' \subset \pi$. Setting $\gamma' = w_\subset^{-1}\alpha$, we obtain

$$\delta' := \frac{1}{2}\left(\gamma' + \sum_{\alpha' \in \pi''} \alpha'\right),$$

to be a sum of roots. Moreover by adding $-\gamma'$ if necessary we can assume that γ' is a positive root. Then this relation is just that occurring in the hypothesis of Lemma 4.7. Through (i) of its conclusion and 4.6, we obtain (i) and (ii) of the present lemma when $r = 1$.

In all other cases we deduce that δ' is a short root. For type G_2 there is little difficulty. Indeed (in the notation of (iii) of Lemma 4.7) γ' is long, α' is short and they are orthogonal. Thus $\gamma' + 3\alpha'$ is a long root. Moreover $w_\subset(\gamma' + 3\alpha') = \alpha + 3\gamma = \varpi_\alpha$. Comparison with our proposed form of θ, this gives (i) and (ii) in that case also.

Suppose now that $r = 2$. Thus Δ is of type B or F. In this case Π' can of course have only one short root.

Observe that since γ' is a long root, possibility (iv) of Lemma 4.7 is excluded.

Set $\delta := w_\subset\delta'$, which equals $\frac{1}{2}(\alpha + |\gamma)|$ for some $\gamma \in \Pi'$, hence is orthogonal to α by 4.6, and is a short positive root. Since ϖ_α is a sum of long roots, it follows by the first paragraph of 4.4 that there exists a short root $\epsilon \in \Pi'$, so orthogonal to δ and unique by remark above, occurring with an integer coefficient $m = \frac{1}{2}\epsilon^\vee(\varpi_\alpha)$ in ϖ_α. Yet in types B, F the sum of two orthogonal simple short roots is always a root (and necessarily long), so $\delta + |\epsilon|$ is the required positive root, necessarily long and has a non-negative scalar product on any $|\gamma| \in \Pi'$, by 4.6.

This completes the proof of (i) and (ii).

Part (iii) of the lemma follows from the remarks in 4.5 concerning the value of h_* on Π and on π. $\square$

4.9. As in 3.5 the non-vanishing of appropriate matrix coefficients together with a degree argument would allow us to show that equality holds in the last part of Lemma 4.7. Thus $\varepsilon_i = 1/2 \Rightarrow h_*(\alpha_i) = -1$. Actually we have the quite remarkable result which is an easy case by case verification. Recall that a root is deemed to be short only if Δ is not simply-laced.

Lemma. *For all $i \in I$, one has*

(i) $h_*(\alpha_i) = 1$, *if and only if α_i is long and $\varepsilon_i = 1$.*
(ii) $h_*(\alpha_i) = 0$, *if and only either α_i is short or $\kappa(\alpha_i) \neq \alpha_i$.*
(iii) $h_*(\alpha_i) = -1$, *if and only if $\varepsilon_i = 1/2$. (In this case $\kappa(\alpha_i) = \alpha_i$.)*

4.10. Through 4.9 the proof of Claim (2) is completed.

To prove Claim (1) we consider first the case when $\alpha \in \pi$ is κ stable. In this case α is short. Moreover we are in types B_{2n}, C_ℓ, F_4. Set $\Pi_{\tau(\alpha)} = \Pi'$.

Recall that $2\varpi_\alpha$ is a sum of elements of $\mathscr{K}$ and hence a sum of long roots by Lemma 4.4. In types B, F the proof is completed as in the last part of 4.8.

Suppose Δ is of type C_ℓ. Then we can choose $\alpha = \alpha_i$: $i = 1, 2, \ldots, \ell - 1$. In the Bourbaki [5] notation,

$$2\varpi_i = \sum_{j=1}^{i} 2\varepsilon_j.$$

In view of 4.6, this forces $\varepsilon_i + \varepsilon_{i+1}$ to occur in $2\varpi_i$. Moreover its sum with α_i is a positive root, that is $2\varepsilon_i$ and equal to $\theta(\alpha_i)$. This completes the proof in type C.

Remark 1. We remark that in type C_ℓ one has $\{2\varepsilon_i\}_{i=1}^{\ell} = \mathscr{K}$. This establishes the claim in 4.2 for type C. Moreover we have shown that up to signs $\varepsilon_i + \varepsilon_{i+i}$: $i = 1, 2, \ldots, \ell - 1$, belong to Π. Since the highest root $2\varepsilon_1$ always belongs to Π, up to a sign, we have determined Π in type C, despite trying to avoid doing so.

Remark 2. The observant reader will notice that in some cases the purported solutions for θ may appear not to exist. However one can easily check that this is only because h_* does not have the correct value. For example in the last part above, the fact that we *cannot* write α_ℓ as a sum of two positive roots with one being simple, is just a reflection of the fact that $h_*(\alpha_\ell) \neq 0$. Again if we go back to Lemma 4.8, then we may remark that $\Pi_{\tau(\alpha_{2i-1})}$ has *no* short root in type B_ℓ. However here $h_*(\alpha_{2i-1}) \neq -1$, so there is again no contradiction.

Summarizing the above and taking account of 4.6 we obtain the

Lemma. *Suppose $h_*(\alpha) = 0$ and $\kappa(\alpha) = \alpha$. Then there is unique $\gamma \in \Pi_{\tau(\alpha)}$ such that $\theta(\alpha) := \alpha + |\gamma|$ is a root. Moreover $\theta(\alpha)$ is a long root and orthogonal to all $\gamma' \in \Pi_{\tau(\alpha)} - \{\gamma\}$.*

4.11. To complete the proof of Claim (1) we must consider the case when $\alpha \in \pi$ is not κ stable. (This occurs in exactly types A_{2n+1}, D_{2n+1}, E_6. In particular Δ is simply-laced and moreover $(\alpha, \kappa(\alpha)) = 0$.)

Fix $\alpha \in \pi$ not κ stable and set $\Pi' = \Pi_{\tau(\alpha)}$.

Evaluating Eq. (6) of 4.3 on $\alpha^\vee$, we conclude that there exists $\gamma \in \Pi'$ such that $\gamma^\vee(\kappa(\varpi_\alpha)) \times \alpha^\vee(\gamma) < 0$. Thus $|\gamma| + \alpha$ is a root and moreover $|\gamma|$ occurs in $\varpi_\alpha + \varpi_{\kappa(\alpha)}$ with a positive integer coefficient.

Add Eqs. (5) and (6) of 4.3. Evaluate the resulting expression on $\gamma^\vee$. Using the orthogonality of the elements of Π' gives

$$\gamma'^\vee\big(\alpha + \kappa(\alpha)\big) = 0, \quad \forall \gamma' \in \Pi'. \tag{$*$}$$

In particular if $\gamma' \in \Pi'$ is κ invariant, it must be orthogonal to α. Thus γ defined above is not κ invariant. Now since c is even, κ fixes both π_a and π_b so in particular commutes with $w_\subset$. Thus κ fixes both Π_a and Π_b. Thus in types D_ℓ: ℓ odd and E_6 both of the two subsets Π_a or Π_b admits at most one non-trivial $\langle\kappa\rangle$ orbit. Moreover in a given orbit both elements either lie in Δ^+ or in Δ^-. Then by $(*)$ only one element $\gamma \in \Pi'$ may be such that $\alpha + |\gamma|$ is a root. We conclude that in these cases $\gamma \in \Pi'$ is uniquely determined exactly as required for Claim (1).

Suppose Δ is of type A_{2n+1}. Then we may choose $\alpha = \alpha_i$, for some $i \in \{1, 2, \dots, n\}$. Then by $(*)$ above it follows that $|\gamma| = \alpha_{i+1} + \cdots + \alpha_{2n+2-i}$, so in particular is uniquely determined exactly as required for Claim (1). Moreover $\beta_i := \alpha + |\gamma| = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{2n+2-i}$ is κ stable. The reader will easily recognize β_i as the ith term in the Kostant cascade. This verifies the last statement of 4.2, for type A_ℓ: ℓ odd.

The proof of Claim (1) is now complete.

Remark 1. Since $\alpha^\vee(|\gamma|) = -1$, so $\kappa(\alpha)^\vee(\kappa(|\gamma|)) = -1$. Uniqueness implies that $\theta(\kappa(\alpha)) = \kappa(\alpha + |\gamma|)$. Again the first relation implies that the roots $\{\alpha\}$, $\Pi_{\tau(\alpha)}$ are linearly independent.

Remark 2. Observe that $(\theta(\alpha), |\gamma'|) = 0$, for all $\gamma' \in \Pi_{\tau(\alpha)} - \{\gamma\}$, via $(*)$ and the uniqueness of γ.

Remark 3. Suppose Δ is of type A_{2n+1}. From the above it follows that $\varepsilon_{i+1} - \varepsilon_{2n+3-i} \in \Pi$, for all $i = 1, 2, \dots, n$, up to signs. Since Π is κ stable and always contains the highest root (up to sign), we have inadvertently determined Π in this case.

4.12. We now consider the case when c is odd and write $c = 2m + 1$ as before. This is exactly type A_{2m}. This case is a little different and perhaps less user-friendly, the cause being the fact that κ interchanges π_a and π_b.

Fix $\alpha \in \pi_a$. Observe that $\Pi_a = w_\subset \pi_a \subset \Delta^-$.

Instead of the expression occurring in the left-hand side of Eq. (7) of 3.6 we consider $w_\subset \sigma_a w_\subset^{-1}$. This equals $\sigma_a w_0 \sigma_a$, if m is even and $\sigma_b w_0 \sigma_b$, if m is odd. If m is even (resp. odd) apply this expression to ϖ_α (resp. $\varpi_{\kappa(\alpha)}$). Noting that σ_a fixes $\varpi_{\kappa(\alpha)}$, this gives

$$\varpi_\alpha + \varpi_{\kappa(\alpha)} = \begin{cases} \sigma_a\kappa(\alpha) + \sum_{\gamma \in \Pi_a} \gamma^\vee(\varpi_\alpha)\gamma: & m \text{ even}, \\ \sigma_b\alpha + \sum_{\gamma \in \Pi_a} \gamma^\vee(\varpi_{\kappa(\alpha)})\gamma: & m \text{ odd}. \end{cases} \tag{1}$$

Let α_c denote the first term on the right-hand side of (1).

Call $\alpha_i \in \pi_a$ a central root if $\kappa(\alpha)^\vee(\alpha)$ is non-zero and hence equal to -1. Recall (2.15) that we have chosen I_a in type A_{2m} to be the set of even positive integers up to $2m$. Then α_m (resp. α_{m+1}) for m is even (resp. m odd) is the unique central root.

One checks that $\alpha_c^\vee(\varpi_\alpha + \varpi_{\kappa(\alpha)}) = 1 - \kappa(\alpha)^\vee(\alpha)$. Thus by (1)

$$\sum_{\gamma \in \Pi_a} (\alpha_c)^\vee(\gamma)\gamma^\vee(\varpi_{\kappa^m(\alpha)}) = \begin{cases} 0: & \alpha \text{ the central root,} \\ -1: & \text{otherwise.} \end{cases} \tag{2}$$

Suppose that α is not the central root.

By (2) there exists $-\gamma \in -\Pi_a \subset \Delta^+$ such that

$$-1 \geqslant (\alpha_c)^\vee(\gamma)\gamma^\vee(\varpi_{\kappa^m(\alpha)}). \tag{3}$$

In fact equality must hold in the above since we are in type A.

We claim that $\gamma \in \Pi_a$ with the above property must be unique and any other element of Π_a must vanish on α_c.

Indeed set $\gamma' := w_\subset^{-1}(\alpha_c)$ and $\alpha' := w_\subset^{-1}(-\gamma) \in \pi_\alpha$. Now if there is more than one root with the above property, then by (2) there must be at least *three* elements of π_a whose scalar product is non-zero on γ'. However by the definition of π_a and since we are in type A this is impossible for any root γ'.

Denote the above unique root as γ_α.

Suppose that $\alpha \in \pi_a$ is the central root. Adopt the convention that $\alpha_i = 0$ for $i > 2m$. Note that one always has $\kappa^m(\alpha) = \alpha_m$.

One checks that $\alpha_c = \alpha_m + \alpha_{m+1} + \alpha_{m+2}$, in all cases. In particular $h_*(\alpha_c) = 1$. On the other hand $h_*(|\gamma|) = 1$, for all $\gamma \in \Pi$. Since we are in type A, it follows that $\alpha_c + |\gamma|$: $\gamma \in \Pi_\alpha$ cannot be a root. Through Eq. (2) it follows that

$$(\alpha_c, \gamma) = 0, \quad \text{or} \quad \gamma^\vee(\varpi_{\kappa^m(\alpha)}) = 0, \quad \forall \gamma \in \Pi_a.$$

On the other hand by Lemma 2.20, there exists $\gamma \in \Pi_a = w_\subset \pi_a$ such that $(\alpha_c, \gamma) \neq 0$. Evaluating both sides of Eq. (1) at $\gamma^\vee$ and using the above we conclude that

$$\gamma^\vee(\alpha_c) = \gamma^\vee(\varpi_{m+1}).$$

Since $h_*(|\gamma|) = 1$ and γ is a negative root, this relation and the above formula for α_c forces $\gamma = -(\alpha_{m+1} + \alpha_{m+2})$. We denote this root by γ_α.

Now we may define $\theta(\alpha)$ to have the analogous properties to those of the case when c is even. First if α is a central root, we set $\theta(\alpha) = \alpha_c$. Otherwise we set $\theta(\alpha) = \alpha_c - \gamma_\alpha$ noting that this root, in the language of Remark 4.7, appears in the right-hand side of (1). One easily checks that h_* takes the value 0 on α_c if α is not the central root. By Corollary 2.17 it follows that $h_*(\theta(\alpha)) = 1$, for all $\alpha \in \pi$.

Thus with the above definitions we have shown the

Lemma. *Suppose c is odd. Then $(\theta(\alpha), |\gamma|) \geqslant 0$, $\forall \gamma \in \Pi_a$, with a strict inequality for a unique $\gamma \in \Pi_a$.*

Remark and Notation. Set $z(\alpha)=\theta(\alpha)-\gamma_\alpha$. One has

$$z(\alpha)=\begin{cases}\alpha_m: & \alpha \text{ the central root},\\ \alpha_c: & \text{otherwise}.\end{cases} \tag{4}$$

Observe that $(z(\alpha),|\gamma|)\leqslant 0$, for all $\gamma\in\Pi_a$. Since the elements of Π_a are pairwise orthogonal, it follows that the $\{z(\alpha),\Pi_a\}$ is a set of linearly independent roots. Note that from (4) and the discussion following (1) it follows that

$$z(\alpha)^\vee(\varpi_\alpha+\varpi_{\kappa(\alpha)})=1. \tag{5}$$

5. Non-vanishing of opposed matrix elements

5.1. Before coming to the main goal of this section we dispense with a few preliminaries. Recall the definition of $\tau:\pi\to\pi_c$ given in 4.3 for c even. We extend its definition to c odd through

$$\tau(\alpha)=\begin{cases}a: & \alpha\in\pi_a,\\ b: & \alpha\in\pi_b.\end{cases}$$

We sometimes write $\tau(i)$ for $\tau(\alpha_i)$, for $i\in I$.

Set

$$sg(i)=\begin{cases}(-1)^c: & i\in I_a,\\ (-1)^{c+1}: & i\in I_b.\end{cases}$$

One checks that

$$\Pi_{\tau(\alpha_i)}\subset sg(i)\Delta^+,\quad \forall i\in I. \tag{$*$}$$

Recall that $\theta(\alpha)$: $\alpha\in\pi$ and $|\gamma|$: $\gamma\in\Pi$ are positive roots, the former being independent of the choice of α in its $\langle\kappa\rangle$ orbit.

Recall 4.12. We extend the definition of z for c even by setting $z(\alpha)=\alpha$, for all $\alpha\in\pi$. Let I_0 be a set of representatives in I of the non-trivial $\langle\kappa\rangle$ orbits in I, with $I_0\subset I_a$, if c is odd. Set $\pi_0:=\{z(\alpha_i)\mid i\in I_0\}$.

Through $sg(i)$ we may define the signs needed in the definition of T introduced in 4.2 for c even, namely

$$T_0:=\{sg(i)\alpha_i\colon i\in I_0\},\qquad T_1:=\{sg(i)\theta(\alpha_i)\}\colon\quad i\in I,\ T:=T_0\sqcup T_1.$$

5.2. Collecting results from 4.6, 4.8(ii), 4.9–4.12, we obtain the

Lemma. *One has $(\theta(\alpha),|\gamma|)\geqslant 0$ for all $\gamma\in\Pi_{\tau(\alpha)}$ and for all $\alpha\in\pi$.*

5.3.

Lemma. *Fix $\alpha\in\pi$. The root vectors $x_{\theta(\alpha)},x_{|\gamma|}$: $\gamma\in\Pi_{\tau(\alpha)}$, commute pairwise.*

Proof. Since $\Pi_{\tau(\alpha)}$ is a W translate of $\pi_{\tau(\alpha)}$, we have only to show that $x_{\theta(\alpha)}$ commutes with the remaining elements. By Lemma 5.2, it suffices to show that $\theta(\alpha)$ and $\gamma \in \Pi_{\tau(\alpha)}$ cannot both be short roots. In this we can assume that Δ is not simply-laced, so then κ fixes any $\alpha \in \pi$. Then $\theta(\alpha)$ is long if $h_*(\alpha) = 0$ (resp. -1) by 4.10 (resp. 4.8(ii)). If $h_*(\alpha) = 1$, then $\theta(\alpha) = \alpha$ and this is a short root only in type B_{2n+1}. However in this case $c/2 = m = 2n + 1$ is odd and so if α is short all the roots in Π_b are long. $\square$

5.4. We now come to the main point of this section. Recall 3.2. Fix $i \in I$ and let $b_{(i)} := b_{\xi,v}$ denote the matrix coefficient in the Hopf dual $U(\mathfrak{g})^\star$, where v is a non-zero vector in the finite dimensional module $V(sg(i)\varpi_i)$ of extremal weight $sg(i)\varpi_i$ and ξ a vector in the dual of extremal weight $sg(i)w_0\varpi_i$. When c is divisible by 4 and $\alpha \in \pi_a$, then this coincides with our previous definition given in 3.4. Unlike our treatment in 3.4 we are now consider all cases simultaneously. The reader may judge whether or not proofs become more transparent!

Let $b_{(i)}^{op}$ denote the matrix coefficient defined by replacing $sg(i)$ by $-sg(i)$ in $b_{(i)}$ above. Through the remarks in (3.1) it follows that $gr_{\mathscr{F}} b_{(i)}$ (resp. $gr_{\mathscr{F}} b_{(i)}^{op}$) is an element of $Y(\mathfrak{n})$ or of $Y(\mathfrak{n}^-)$, depending on signs, and is the ε_i^{-1} power of a generator. Set $k_i = \deg gr_{\mathscr{F}} b_{(i)}$, which we note coincides with our previous definition in 3.4, given in the special case, c divisible by 4 and $i \in I_\alpha$.

To motivate the result we wish to prove, we let τ^{op} denote the opposite of τ, that is

$$\tau^{op}(\alpha) = \begin{cases} a\colon & \tau(\alpha) = b, \\ b\colon & \tau(\alpha) = a. \end{cases}$$

(Recall $s(i)$: $i \in I$ defined in 3.2. One may remark that for c even one has $\Pi_{\tau^{op}(\alpha_i)} \subset s(i)\Delta^+$. Thus in view of $(*)$ of 5.1 we may consider $s(i)$ as $sg(i)^{op} = -sg(i)$.)

Let x be a sum of root vectors. Then by a monomial occurring in a power of x we mean a product of root vectors occurring in the development of x. If these root vectors commute and the roots in question are linearly independent (which will be the case here), then this monomial is determined by its weight.

Recalling the notation of 3.4 one may observe that in 3.4, 3.6, we proved that there exists a unique monomial y_i occurring in $y_{\tau^{op}(\alpha_i)}^{k_i}$ such that $b_{(i)}(y_i) \neq 0$.

Proposition. *For all $i \in I$, there exists a unique monomial y_i^{op} occurring in*

$$x_{sg(i)\theta(\alpha_i)}^{1/\varepsilon_i} y_{\tau(\alpha_i)}^{k_i - 1/\varepsilon_i}$$

such that $b_{(i)}^{op}(y_i^{op}) \neq 0$.

The proof of this result will be given in the subsequent sections. For the moment we just make three remarks. Firstly $\{sg(i)\theta(\alpha_i)\}$ and the roots in $\Pi_{\tau(\alpha_i)}$ are either all positive or all negative and their sum defined by 4.3, Eq. (6), equals $-\operatorname{wt} b_{(i)}^{op}$. Secondly these monomials all total have degree k_i and degree ε_i^{-1} in the elements of $x_{sg(i)\theta(\alpha_i)}$. Thirdly by 5.3 the root vectors in the above product, commute and are linearly independent by 4.6, 4.10 and Remark 4.12. This implies uniqueness.

5.5. In what follows it is only the notation that is formidable. Indeed the proof of the Proposition 5.4 is exactly that of Proposition 3.4 in almost all cases. To be precise these cases are exactly those for which equality holds in the conclusion of Lemma 5.2. The key point is that we start from a highest (or lowest) weight vectors in an appropriate simple module and then the powers of each root vector take extremal vectors to extremal vectors. Of course weights must all match up but this is just by orthogonality as in the proof of Proposition 3.4.

5.6. Fix $\alpha \in \pi$. We say that orthogonality fails (relative to π), if there exists $\gamma \in \Pi_{\tau(\alpha)}$ such that $\gamma^\vee(\kappa(\alpha)) \neq 0$ and $(\theta(\alpha), \gamma) \neq 0$, and orthogonality holds otherwise.

Orthogonality can fail if the coefficient of $\kappa(\alpha)$ in some $\Pi_{\tau(\alpha)}$ is too large. When $\kappa(\alpha) \neq \alpha$, there is just one (!) such case occurring in type E_6. In this case treated below we just need to be a little careful about the order we apply the monomials to extremal weights of $V(-sg(i)\varpi_i)$ (even though the commute) because we want weights to remain extremal after acting by a given power of a root vector in order to be sure of non-vanishing. We call this a good ordering. We treat this case in 5.8 without any particular reference to E_6 or whether orthogonality fails. However it was perhaps worthwhile to have explained why we need to go to this extra trouble!

Otherwise there are just five exceptions (occurring in types F_4, E_7, E_8). These cases do not admit a good ordering. To handle them we shall need a little extra theory. As we note in the Appendix there are in fact only 6 exceptional cases all occurring in the exceptional Lie algebras, so this may seem like much pain for little gain. However at least we do not have to list the exceptions nor compute Π, which we nevertheless do in the Appendix for the benefit of the inquisitive reader.

5.7.

Lemma. *Assume $\kappa(\alpha) \neq \alpha$. Orthogonality fails if and only if Δ is simply-laced, c is even and the coefficient of α in $\theta(\alpha)$ is > 1.*

Proof. By 4.6, 4.9 and 4.12, Eq. (4) we can assume $h_*(\alpha) = 0$. Suppose c is even. Then orthogonality fails exactly when the coefficient of $\kappa(\alpha)$ in the unique γ occurring in Claim (1) of 4.5 is > 1. When c is odd a similar argument shows that orthogonality cannot fail (via 4.12 and because we are in type A_{2m}). □

5.8. Under the hypotheses of the above lemma, orthogonality fails only if Δ is simply-laced and c is even. Fix $\alpha \in \pi$ which is not κ invariant and recalling Remark 2 of 4.11, let γ be the unique element of $\Pi_{\tau(\alpha)}$ such that $\alpha^\vee(\gamma) \neq 0$. To lighten notation we assume that γ is a positive root. Set $\alpha' = \kappa(\alpha)$, $\gamma' = \kappa(\gamma)$ and $\varpi = \varpi_\alpha$. Let $t + 1$ be the coefficient of α' in γ (which is also the coefficient of α in γ'). Thus by Remark 1 of 4.11, we have $t \geqslant 0$. Moreover t is also the coefficient of α' in γ', or of α in γ. In fact $t \leqslant 1$, though we do not need to know this. Set $\Pi'' = \Pi_{\tau(\alpha)} - \{\gamma, \gamma'\}$. Then Eq. (6) of 4.3 becomes

$$\varpi + \kappa(\varpi) = \theta(\alpha) + t\big(\gamma + \gamma'\big) + \sum_{\gamma'' \in \Pi''} \gamma''^\vee\big(\kappa(\varpi)\big)\gamma''. \tag{1}$$

Lemma. *The following*

(i) $x^t_{-\gamma'} x_{-\theta(\alpha)} x^t_{-\gamma} v_\varpi$,
(ii) $x^t_{-\gamma'} x^{t+1}_{-\gamma} x_{-\alpha} v_\varpi$,

are non-zero extremal vectors.

Proof. This is just a matter of successively checking that the exponent s of a given x_δ: $\delta \in \Delta^-$ equals the value of $\delta^\vee$ on the weight of the vector to which x_δ^s is to be applied.

Remark. By the orthogonality relations (cf. Remark 2 of 4.11) of the remaining γ'', the conclusion of the lemma continues to hold when we apply the powers of $x_{-\gamma''}$, to the coefficients appearing the right-hand side of Eq. (1) . Thus assertion (i) achieves the goal outlined in the first paragraph of 5.6. Assertion (ii) is needed for Proposition 6.1. Of course these assertions also hold with all signs reversed. □

5.9. Now fix $\alpha \in \pi$ which is κ stable. It is clear from 4.6 and 4.9 that orthogonality failure only arises when $h_*(\alpha) \in \{0, -1\}$. The first case (of which there is only one (!) instance) is handled by a (rather trivial) property of modules in type C_2. The second case (of which there are four instances) is handled by a (very slightly less trivial) property of modules in type D_4. These are described in the following two sections.

5.10. We consider the case of orthogonality failure for $\alpha_i \in \pi$ with $h_*(\alpha_i) = 0$. Then $\theta(\alpha_i) = \alpha_i + |\gamma|$, for a canonically determined $\gamma \in \Pi_{\tau(\alpha_i)}$. Since by 4.6 one has $(\theta(\alpha_i), \gamma') = 0$, for all $\gamma' \in \Pi_{\tau(\alpha_i)} - \{\gamma\}$ it is enough to restrict to the pair $\{\theta(\alpha), \gamma\}$ in establishing Proposition 5.4, that is to say one may reduce to the following situation.

Take $\pi = \{\alpha, \beta\}$ of type C_2 with α short. Let s, t be non-negative integers and let ϖ be the dominant weight defined by $\alpha^\vee(\varpi) = s$, $\beta^\vee(\varpi) = t$. Let $v_\varpi, v_{-\varpi}$ be the highest and lowest weight vectors in $V(\varpi)$, defined of course relative to π'. Obviously

$$x_{-\alpha}^s x_{-(\alpha+\beta)}^{s+2t} v_\varpi = * v_{-\varpi}, \tag{1}$$

where here and in the next section $*$ denotes a non-zero scalar.

Lemma.

$$x_{-(\alpha+\beta)}^{2t} x_{-(2\alpha+\beta)}^s v_\varpi = * v_{-\varpi}. \tag{2}$$

Proof. One has

$$x_{(\alpha+\beta)}^s x_{-(2\alpha+\beta)}^s v_\varpi = * x_{-\alpha}^s v_\varpi \neq 0.$$

This means that $x_{-(2\alpha+\beta)}^s v_\varpi$ has a non-zero projection onto the unique direct summand of the $\mathfrak{sl}(2)$ submodule

$$\bigoplus_{n \in \mathbb{N}} V(\varpi)_{\varpi - s\alpha - n(\alpha+\beta)},$$

of $V(\varpi)$ of highest weight $\varpi - s\alpha$. Yet $(\alpha + \beta)^\vee(\varpi - s(2\alpha + \beta)) = 2t - s$. Hence the assertion of the lemma. □

Remark. Note that Eq. (2) has a smaller overall exponent than Eq. (1). This is the essence of our replacement of α by $\theta(\alpha)$, which in the case, $\kappa(\alpha) = \alpha$ and $h_*(\alpha) = 0$, are both short roots with relative root lengths $r = 2$.

Applying the lemma with $\alpha = \alpha_1, \gamma = \alpha_1 + \alpha_2, k_1 = 1$ and k_2 by the specifics of the choice of root system and the choice of $i \in I$, results in a proof of the proposition when $h_*(\alpha_i) = 0$ and α_i is κ stable. Notice here that $\theta(\alpha_i)$ becomes $2\alpha + \beta$.

We remark that when orthogonality holds (that is when $(\theta(\alpha), \gamma') = 0$, for all $\gamma' \in \Pi_{\tau(\alpha)}\}$) one takes $k_2 = 0$, whilst for the one example in F_4, we must take $k_2 = 1$, though we do not need to know this last fact.

5.11. We consider the case of orthogonality failure for $\alpha \in \pi$ with $h_*(\alpha) = -1$. Then $\theta(\alpha) = \alpha + \gamma_1 + \gamma_3 + \gamma_4$, for canonically determined $\gamma_1, \gamma_3, \gamma_4 \in \Pi_{\tau(\alpha_i)}$. Since by 4.6 one has $(\theta(\alpha), \gamma') = 0$, for all $\gamma' \in \Pi_{\tau(\alpha)} - \{\gamma_1, \gamma_3, \gamma_4\}$ it is enough to restrict to the quartet $\{\theta(\alpha), \gamma_1, \gamma_3, \gamma_4\}$ in establishing Proposition 5.4, that is to say one may reduce to the following situation.

Take $\pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of type D_4 in the Bourbaki [5] notation. In particular α_2 is its unique trivalent root α_*. Again $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ is the unique highest root β_*, relative to π'. Observe that $\alpha_0 := \frac{1}{2}(\beta_* + \alpha_1 + \alpha_3 + \alpha_4) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, is a root. Let s_1, s_3, s_4 be non-negative integers and set $s_0 = 1$. Set $\varpi = \sum_{i=0,1,3,4} s_i \alpha_i = \frac{1}{2}\beta_* + \sum_{i=1,3,4}(s_i + \frac{1}{2})\alpha_i$. Though ϖ is not dominant, nevertheless there is a unique up to isomorphism finite dimensional module $V(\varpi)$ relative to π' of extremal weight ϖ. Let $v_\varpi, v_{-\varpi}$ be extremal weight vectors in $V(\varpi)$, of the given weight. Obviously

$$x_{\beta_*} \prod_{i=1,3,4} x_{-\alpha_i}^{2s_i+1} v_\varpi = * v_{-\varpi}, \tag{1}$$

where we recall that here $*$ denotes a non-zero scalar.

Lemma.

$$\prod_{i=0,1,3,4} x_{-\alpha_i}^{2s_i} v_\varpi = * v_{-\varpi}. \tag{2}$$

Proof. Recall that if ϖ is an extremal weight of a finite dimensional simple module and γ is a root satisfying $\gamma^\vee(\varpi) \leqslant 0$, then $(\gamma + \varpi, \gamma + \varpi) > (\varpi, \varpi)$ and so $x_\gamma v_\varpi = 0$. Observe that

$$(\alpha_1 + \alpha_2)^\vee(\varpi) = (s_1 - s_3 - s_4), \qquad (\alpha_2 + \alpha_3 + \alpha_4)^\vee(\varpi) = 1 - (s_1 - s_3 - s_4).$$

Thus either $x_{\alpha_1+\alpha_2} v_\varpi$ or $x_{\alpha_2+\alpha_3+\alpha} v_\varpi$ must vanish. The same applies to the corresponding expressions with $1, 3, 4$ cyclically permuted. We conclude that

$$x_{\alpha_1} x_{\alpha_3} x_{\alpha_4} x_{-\alpha_0}^2 v_\varpi = x_{-\beta_*} v_\varpi \neq 0, \tag{3}$$

whilst this non-zero expression is annihilated by all the α_i: $i = 1, 3, 4$. Thus as 5.10, $x_{-\alpha_0}^2 v_\varpi$ has a non-zero projection onto the $\mathfrak{sl}(2)^3$ submodule of $V(\varpi)$ whose weights lie in $\varpi - \beta_* - \mathbb{N}\alpha_1 - \mathbb{N}\alpha_3 - \mathbb{N}\alpha_4$. On the other hand, $\alpha_i^\vee(\varpi - 2\alpha_0) = 2s_i - 1$: $i = 1, 3, 4$, hence the assertion of the lemma.

This result is applied with $\alpha = \alpha_0$ and $\gamma_i = \alpha_i$: $i = 1, 3, 4$. Then $\theta(\alpha)$ is the highest root in D_4 above. The choice of s_1, s_3, s_4 is dictated by the particular root system and the choice of α. The trivial case is when orthogonality holds and then the s_i: $i = 1, 3, 4$ are all taken equal to zero.

The most extreme case is in type E_8 with α_* the trivalent root. In this case we need $s_1 = s_3 = 1$, $s_4 = 2$, though of course we do not need to know this. $\square$

This completes the proof of Proposition 5.4.

5.12. Recall the set Λ_0 defined in 3.1. Take $\mu \in \Lambda_0$. We remark that if $a_\mu(y) \neq 0$ then $a_{-\mu}(y) = 0$ and vice-versa.

This follows from the form of y and the linear independence of the elements of Π. Recalling T_0, T_1, T defined in 5.1 we let V_0, V_1, V denote the linear subspace of $\mathfrak{g}$ spanned by the root vectors whose roots lie in the corresponding sets. Now Proposition 3.4 was deduced from Eq. (6) of 3.4 and a comparison of degrees using h_*. Recalling that $h_*(\theta(\alpha)) = 1$, for all $\alpha \in \pi$, we similarly obtain from Proposition 5.4 the following

Theorem. *Fix $\mu \in \Lambda_0$. If $a_\mu(y) = 0$, then $a_\mu(y + V_1) \neq 0$, with a similar assertion when μ is replaced by $-\mu$.*

Remark 1. We now have the following beautiful explanation of ε_α. Assume for example that $a_\mu(y) = 0$, with $\mu = 2\varepsilon_\alpha \varpi_\alpha$. Then identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form, it follows that $a_\mu(y + V_1)$ is just the linear function $x_{\theta(\alpha)}$, whereas had we made the evaluation forgetting (that is ignoring the existence of) ε_α we would have ended up with the possibly quadratic function $x_{\theta(\alpha)}^{1/\varepsilon_\alpha}$. Of course even after taking care of this adjustment, it is quite miraculous that we do end up with a linear function.

Remark 2. Take $\mu = \varepsilon_\alpha(\varpi_\alpha + \kappa(\varpi_\alpha))$. Suppose $a_\mu(y) \neq 0$. Then the value of a_μ on y coincides with its value on the whole of $y + V$, that is to say the elements of V make no further contribution. This follows from the linear independence of the elements of $\{\alpha\} \cup \Pi_{\tau(\alpha)}$.

6. The remaining contributions from the semicentre

6.1. To set up our metaslice theorem we still have to consider the possible contribution from $Sy(\mathfrak{b})$, or from $Sy(\mathfrak{b}^-)$. From [18, 4.14] we know that the number of additional generators is just $\operatorname{card}\pi_0$, in the notation of 4.2. As discussed in [12], these additional generators are obtained as follows. First given $\alpha_i \in \pi$ we have $\kappa(\alpha_i) \neq \alpha_i$. Consequently we have two matrix coefficients $b_{(i)}$ and $b'_{(i)}$, with the first defined as in 5.4 and the second replacing α_i by its κ translate. Let us assume $sg(i) = 1$, as the case $sg(i) = -1$ will be exactly the same. Then both the above elements have diagonal weight $\varpi_i + \kappa(\varpi_i)$.

Yet both $gr_{\mathscr{F}} b_{(i)}$ and $gr_{\mathscr{F}} b'_{(i)}$ are non-zero elements of $Y(\mathfrak{n})$ having weight $\varpi_i + \kappa(\varpi_i)$. Now the latter is one-dimensional and this forces $gr_{\mathscr{F}} b_{(i)} = gr_{\mathscr{F}} b'_{(i)} = a_{\varpi_i + \kappa(\varpi_i)}$, up to non-zero scalars. Consequently $gr_{\mathscr{F}} b''_{(i)}$, where $b''_{(i)} := (b_{(i)} - b'_{(i)})$, is an ad $\mathfrak{n}$ invariant element of strictly higher degree and by [13, 6.7] must belong to $Sy(\mathfrak{b})$ and of course have weight $\varpi_i + \kappa(\varpi_i)$. By [18, 4.16], it must therefore coincide with the additional generator of $Sy(\mathfrak{b})$ of the above weight and hence have degree equal 1+ deg $gr_{\mathfrak{F}} b_{(i)}$. (This degree result can also be obtained from the Proposition below.) Recall the definition of z given in 4.12, 5.1. In particular it is identity if c is even.

Following the above we define $b''^{op}_{(i)}$, by replacing $sg(i)$ by $-sg(i)$.

Proposition. *Take $\alpha_i \in \pi$ with $\kappa(\alpha_i) \neq \alpha_i$ and $i \in I_a$ if c is odd. Then with $b''_{(i)}$, defined as above*

$$b''^{op}_{(i)}\big(y^{k_i}_{\tau(\alpha_i)} x_{sg(i)z(\alpha_i)}\big) \neq 0.$$

Proof. To concentrate ideas we assume that c is even. As before we may just treat the case $sg(i) = 1$ to lighten notation, though in fact for c odd one always has $sg(i) = -1$.

Since α_i is not κ invariant, we must have $\varepsilon_i = 1$ as noted in 3.1. Thus by Proposition 5.4, and the present choices we must have

$$b^{op}_{(i)}\big(y^{k_i-1}_{\tau(\alpha_i)} x_{\theta(\alpha_i)}\big) \neq 0, \tag{1}$$

where we have profited from 5.3 to reverse order. (This is just a convenience.)

Similarly

$$b'^{op}_{(i)}\big(y^{k_i-1}_{\tau(\alpha_i)} x_{\theta(\alpha_i)}\big) \neq 0. \tag{2}$$

We denote α_i simply as α. Recall that $\theta(\alpha) = \alpha + |\gamma|$, for some uniquely determined $\gamma \in \Pi_{\tau(\alpha)}$. We can therefore write

$$x_{\theta(\alpha)} = x_\alpha x_{|\gamma|} - x_{|\gamma|} x_\alpha. \tag{3}$$

Replace in Eqs. (1), (2) above, the left-hand side of (3) by the second term in its right-hand side. Since $x_\alpha v_{-\kappa(\varpi_\alpha)} = 0$, the left-hand side of (2) becomes zero. We claim that conversely the left-hand side of (1) does not become zero. Since c is even, this is just (ii) of 5.8. Finally we may absorb the factor of $x_{|\gamma|}$ by increasing the exponent of $y_{\tau(\alpha)}$ by one.

For c odd, there is an overall sign change which we shall ignore. Then the only slight difference is that $\theta(\alpha) = z(\alpha) + |\gamma|$, so $x_{z(\alpha)}$ replaces x_α in (3). Replace in Eq. (1) (resp. (2)), the left-hand side of (3) by the terms in its right-hand side. We claim that either the first term gives a zero contribution to (2), and the second term gives a zero contribution (3) or vice-versa. Since here orthogonality holds by 5.7, to prove this claim, we may take $x_{z(\alpha)}$ to the extreme right (for the first term) or to extreme left (for the second term), and use the fact that noted in Remark 4.12 that exactly either $z(\alpha)^\vee(\varpi_\alpha)$ or $z(\alpha)^\vee(\varpi_{\kappa(\alpha)})$ equals zero. □

6.2. Let ς be the Chevalley antiautomorphism given by $\varsigma(x_\alpha) = x_{-\alpha}$, for all $\alpha \in \Delta$. Recall that the $\mu_i = \varepsilon_i(\varpi + \kappa(\varpi))$: $i \in I$ generate Λ_0. If c is even μ_i depends only on the $\langle\kappa\rangle$ orbit of α_i. If c is odd we always choose $i \in I_a$. Set

$$\varsigma_i = \begin{cases} Id: & sg(i) = -1, \\ \varsigma: & sg(i) = 1, \end{cases}$$

and

$$X_i := \varsigma_i\big(Y(\mathfrak{n}^-)_{-\mu_i} Sy(\mathfrak{b})_{\mu_i}\big).$$

As noted in 6.1, $Sy(\mathfrak{b})_\mu$ is strictly larger than $Y(\mathfrak{n})_\mu$ and is two-dimensional if and only if ϖ_i is not fixed by κ. Thus if ϖ_i is κ fixed we simply have

$$X_i = Y(\mathfrak{n}^-)_{-\mu_i} Y(\mathfrak{n})_{\mu_i},$$

which is furthermore one-dimensional.

Let $A(\mathfrak{g})^{sg}$ be the subalgebra of $S(\mathfrak{g})$ generated by the set of X_i defined above. Recall the definition of V given through 5.1, 5.12. Combining 5.12 and 6.1 gives a version of our metaslice

Theorem. *Restriction of functions gives an isomorphism of $A(\mathfrak{g})^{sg}$ onto the space $R[y+V]$ of regular functions on $y+V$.*

Remark. If $-1 \in W$, then $A(\mathfrak{g})^{sg} = A(\mathfrak{g})$ as subalgebras of $S(\mathfrak{g})$. Otherwise they may be very slightly different (though isomorphic as algebras). The result announced in the introduction is given in 8.8.

7. Slices

7.1. Before going further it seems appropriate to discuss the various notions of a slice which have been introduced in the literature [26,28,38,22].

Let A be a connected algebraic group acting regularly on an irreducible algebraic variety X, that is the resulting map $A \times X \longrightarrow X$ is a morphism of algebraic varieties. If Y is a subvariety of X and $y \in Y$, we denote by $T_{y,Y}$ the tangent space in Y at y. Then we say that a subvariety Z of X cuts Y transversely at $y \in Y$ if $y \in Z$ and $T_{y,Y} \cap T_{y,Z} = 0$. One has $\dim T_{y,Ay} = \dim Ay$.

To the present author the natural definition of a slice to A orbits in X should involve an appropriate subset $\mathscr{S}$ of X such that each A orbit meets $\mathscr{S}$ at exactly one point. In this a picturesque example arises when X is the two-sphere and $A = SO(2)$ acting by rotations about a fixed axis. Then the longitudes form the orbits and any latitude is a slice.

In general it is too much to expect all orbits to pass through $\mathscr{S}$. Moreover in the above example the latitudes are not (even real) algebraic varieties, whilst great circles through the poles meet each longitude twice. Nevertheless this example indicates that a slice will not be unique and cannot be expected to cut (and by this we mean transversally) through each orbit.

A better example is the following. Let $\mathbf{k}$ be a field and n a positive integer. Set $E(n,\mathbf{k}) = \mathbf{k}^n$, $M(n,\mathbf{k}) = End_{\mathbf{k}}\mathbf{k}^n$ and $GL(n,\mathbf{k})$ its subgroup of invertible elements. (We may omit $\mathbf{k}$ and/or n below.) The natural action of G on E induces an action on M through conjugation. Fix a basis in E and let $x_{i,j}$: $i,j = 1,2,\dots,n$ denote the basis of M of elementary matrices. Let Ω be the subset of $M(n)$ of matrices admitting a cyclic vector. It is well known that the subset $C := y+V$, where $y = \sum_{i=1}^{n-1} x_{i+1,i}$, $V := \bigoplus_{i=1}^{n} \mathbb{C}x_{i,n}$, of "companion" matrices, meets every G orbit in Ω at just one point. (We call this the companion slice.) Indeed if $a \in \Omega$ and v is a cyclic a vector, then $a^{i-1}v$: $i = 1,2,\dots,n$ is a basis for E with respect to which a lies in C. Then the assertion follows from the fact that G acts transitively on the set of all basis of E with trivial stabilizer.

The above example has been extended to four further sets of cases (when the base field $\mathbf{k}$ is algebraically closed of characteristic zero which we take to be the complex field $\mathbb{C}$ for simplicity of presentation).

In the first case replace $GL(n)$ by $SL(n)$ and $M(n)$ by the subspace of matrices of trace zero. Then conjugation may be viewed as coadjoint action of $A = SL(n)$ on $(\operatorname{Lie} A)^*$. Now further replace A by a complex connected simple algebraic group G and identify $\mathfrak{g} := \operatorname{Lie} G$ with its dual $\mathfrak{g}^*$ through the Killing form. Following Kostant [27] fix a principal s-triple (x,h,y) in $\mathfrak{g}$. Then [27, Thm. 0.10] the affine translate $y+\mathfrak{g}^x$ meets every G orbit in $\mathfrak{g}^*_{reg}$ in exactly one point. We call $y+\mathfrak{g}^x$ the Kostant slice. Again $G(y+\mathfrak{g}^x)$ is irreducible and coincides with the regular sheet in the sense of Dixmier [9].

The second case is when $\mathfrak{g}$ is simple and X a finite dimensional simple $\mathfrak{g}$ module such that the invariant algebra $R[X]^{\mathfrak{g}}$ is polynomial (which is very rare outside the coadjoint representation). This result was obtained in [33] basically by case by case considerations – see also [34, 2.2.10]. Popov calls these Weierstrass sections since the first case was obtained by Weierstrass who considered the problem of obtaining a normal form for non-singular elliptic curves. This corresponds to taking $\mathfrak{g} = \mathfrak{sl}(3)$ with X one of its ten-dimensional simple module representing the space of cubic 3-forms. (The corresponding weight diagram is somewhat ubiquitous. It occurs as the Tetractys in a Pythagorean cult, as the Tetragrammaton in Kabbalistic tradition, as the multiplet of elementary particles which predicted the Ω^- and from the sublime to the ridiculous – in ten-pin bowling.)

For the third case, let $\mathfrak{q}$ be a truncated biparabolic subalgebra of $\mathfrak{sl}(n, \mathbb{C})$. Then by [22] there exists an affine translate $y + V$ in $\mathfrak{q}^*_{reg}$ meeting regular Q orbits in exactly one point. However in this case, although $Q(y + V)$ is dense in $\mathfrak{q}^*$, it does not usually equal $\mathfrak{q}^*_{reg}$.

For the fourth case one may take centralizers [23] in type $\mathfrak{sl}(n)$.

7.2. Let $\mathfrak{g}$ be a complex simple Lie algebra. Some years after Kostant's work the notion of a slice was modified by Luna [28]. Later Slodowy [38] showed that this notion could encompass the affine space $y + \mathfrak{g}^x$ for *any* s-triple (x, h, y).

The notion of a slice in the sense of Luna and Slowody involves the notion of a smooth morphism introduced and developed by Grothendieck et al. [15,16]. As we shall use this notion also we briefly cite its main properties in just sufficient generality for our purposes.

Let $\varphi : Y \to Z$ be a morphism of affine algebraic varieties over a field $\mathbf{k}$ and $\varphi^* : R[Z] \to R[Y]$ the corresponding comorphism of algebras of regular functions (which are finitely generated and commutative). Let $\mathbf{m}_y$ be the maximal ideal of $R[Y]$ at $y \in Y$ and φ^*_y be the resulting homomorphism of $R[Z]_{\mathbf{m}_{\varphi(y)}}$ into $R[Y]_{\mathbf{m}_y}$. Suppose $z \in Z$. Then $\mathbf{m}_z$ is the kernel of the composed map $R[Z] \to R[Y] \to R[\varphi^{-1}(z)]$. This makes $R[\varphi^{-1}(z)]$ a $\mathbf{k}(z) := R[Z]/\mathbf{m}_z$ algebra.

In view of [16, Thm. 17.5.1] one may define φ to be smooth at $y \in Y$ if φ is a flat at y (that is to say φ^*_y is flat algebra homomorphism) and $\varphi^{-1}\varphi(y)$ is smooth over $\mathbf{k}(y)$ (that is to say $R[\varphi^{-1}\varphi(y)]$ is a regular (commutative) algebra over a separable extension of $\mathbf{k}(\varphi(y))$. Then φ is said to be smooth if it is smooth at all $y \in Y$. In particular if φ is smooth then φ^* is a flat algebra homomorphism. Then by [16, 2.4.6], $\varphi(Y)$ is open in Z. Moreover if φ is surjective, then φ^* is a faithfully flat embedding.

If Z is a smooth variety, then φ is smooth at $y \in Y$ if the induced map of tangent spaces $d_y\varphi : T_{y,Y} \to T_{f(y),Z}$ is surjective, by [16, 17.11.1].

Henceforth we shall assume that the base field $\mathbf{k}$ is algebraically closed.

Let A be a connected affine algebraic group acting regularly on an irreducible affine algebraic variety X. Then in [38, II,5.1] a transverse slice to the orbit through $y \in X$ is defined to be a locally closed subvariety $\mathscr{S}$ of X such that

(1) $y \in \mathscr{S}$.
(2) The morphism $A \times \mathscr{S} \longrightarrow X$ defined by the action is smooth.
(3) $\dim \mathscr{S}$ is minimal with respect to (1) and (2).

A basic fact [38, p. 60, Lemma 1] is that if X is smooth such a slice always exists. However it is a local notion and not quite what we want here mainly because a given orbit may pass many times through $\mathscr{S}$.

If $x \in X_{reg}$, that is $\operatorname{codim}_X Ax$ takes its minimal value which we shall denote by $\iota_{X,A}$, then one may cut down $\mathscr{S}$ to lie in X_{reg}. However this still does not ensure that a given orbit passes just once through $\mathscr{S}$. For example take G to be a complex connected simple algebraic group and choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} := \operatorname{Lie} G$, identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form. Take $h \in \mathfrak{h}_{reg}$. Then $\mathfrak{h}_{reg}$ is an open neighbourhood of h in $\mathfrak{h}$, whilst $\mathfrak{h}$ is a complement in the tangent space $T_{h'}(\mathfrak{g})$ to the image of $\mathfrak{g}$ at any point $h' \in \mathfrak{h}_{reg}$. In particular $T_{h,Gh} + T_{h,\mathfrak{h}_{reg}} = T_{h,\mathfrak{g}}$. Since $\mathfrak{g}$ is smooth, it follows that $\mathfrak{h}_{reg}$ is a transverse slice to Gh through h. Moreover in this case every G orbit through $\mathfrak{h}_{reg}$ cuts transversally. On the other hand passes $|W|$ times through $\mathfrak{h}_{reg}$.

Thus briefly speaking a slice in the above sense does not even recover Chevalley's theorem, let alone the stronger result of Kostant noted in 7.1.

In the case of a Slodowy slice $\mathscr{S} := y + \mathfrak{g}^x$, one has $T_{y,\mathscr{S}} = \mathfrak{g}^x$ and $T_{y,Gy} = [\mathfrak{g}, y]$, whilst $\mathfrak{g}^x + [\mathfrak{g}, y] = \mathfrak{g}$ and is a direct sum by $\mathfrak{sl}(2)$ theory (as is well known). Thus $\mathscr{S}$ cuts Gy transversally at y. If $z \in \mathscr{S}$ is arbitrary, then $\mathfrak{g}^x + [\mathfrak{g}, z] = \mathfrak{g}$ (by a standard deformation argument – see 7.8, for example) but the sum is usually not direct unless (x, h, y) is a principal s-triple, that is unless $\mathscr{S}$ is the Kostant slice. Whilst the morphism $G \times \mathscr{S} \to \mathfrak{g}$ is smooth, $\mathscr{S}$ need only be cut transversally at y by an orbit.

7.3. Let A be a connected affine algebraic group acting regularly on an smooth affine irreducible algebraic variety X.

In the present work we prefer to define a slice $\mathscr{S}$ to the action of A on X to be a locally closed subvariety of X such that the image of the morphism $A\mathscr{S} \to X$ is dense in X (so $A \times \mathscr{S}$ contains an open subset of X) and such that every orbit in $A\mathscr{S}$ cuts $\mathscr{S}$ transversally at exactly one point.

Recall the notation of 7.2 and set $\mathscr{S}_{reg} := \mathscr{S} \cap X_{reg}$, which is again locally closed in X. Moreover $\overline{A\mathscr{S}_{reg}} = \overline{A\mathscr{S}} = X$, so $\dim \mathscr{S}_{reg} = \iota_{X,A} = \operatorname{codim}_X Ay$: $y \in X_{reg}$. By tranversality, $T_{y,\mathscr{S}} \cap T_{y,Ay} = 0$, at each point $y \in \mathscr{S}$. This and dimensionality implies $T_{y,\mathscr{S}_{reg}} + T_{y,Ay} = T_{y,X}$, at each point $y \in \mathscr{S}_{reg}$. Now the left-hand side is just the image of $T_{(e,y),A\times\mathscr{S}_{reg}}$ in $T_{y,X}$. Since X is smooth it follows that $A \times \mathscr{S}_{reg} \to X$ is a smooth morphism. In particular its image $A\mathscr{S}_{reg}$ is open (dense) in X.

Set $R = R[AS_{reg}]$, $S = R[X]$. Restriction of functions gives an embedding of $R[X]$ into $R[A\mathscr{S}_{reg}]$. Set $D = X - A\mathscr{S}_{reg}$, which is closed and A stable. If $\operatorname{codim}_X D = 1$, let D_1 denote the union of its irreducible components of codimension 1. By the Krull lemma the ideal of definition $I(D_1)$ is principal, that is $I(D_1) = dR[X]$ for some $d \in R[X]$. Then

$$R[A\mathscr{S}_{reg}] \xleftarrow{\sim} \begin{cases} R[X]: & \operatorname{codim}_X D \geqslant 2, \\ R[X][d^{-1}]: & \operatorname{codim}_X D = 1. \end{cases} \tag{1}$$

Again $D' := \mathscr{S} - \mathscr{S}_{reg}$ is closed in $\mathscr{S}$. If $\operatorname{codim}_X D' = 1$, then define $d' \in R[\mathscr{S}]$, in a similar manner to d above. This gives

$$R[\mathscr{S}_{reg}] \xleftarrow{\sim} \begin{cases} R[\mathscr{S}]: & \operatorname{codim}_{\mathscr{S}} D' \geqslant 2, \\ R[\mathscr{S}][d'^{-1}]: & \operatorname{codim}_{\mathscr{S}} D' = 1. \end{cases} \tag{2}$$

Finally since the morphism $A \times \mathscr{S}_{reg} \to X$ is smooth and A orbits can meet $\mathscr{S}$ at just one point, a result (12.3) of Hinich implies that the restriction map gives an isomorphism

$$R[A\mathscr{S}_{reg}]^A \xrightarrow{\sim} R[\mathscr{S}_{reg}]. \tag{3}$$

It is clear that if both $\operatorname{codim}_X D \geqslant 2$ and $\operatorname{codim}_{\mathscr{S}} D' \geqslant 2$, Eqs. (1)–(3) imply

$$R[X]^A \xrightarrow{\sim} R[\mathscr{S}_{reg}]. \tag{4}$$

Remark. Recall that $A\mathscr{S}_{reg}$ is open dense in X and that by hypothesis X is a smooth, hence normal variety. Then (a special case of) a result of Popov [32] asserts that a regular function on $\mathscr{S}$ extends uniquely to a rational invariant function on X which is regular on some open set of X. By the above we can take this open set to be $A\mathscr{S}_{reg}$, which we may compare with [32, Thm. 2]. In order to obtain regularity on all of X we need that the open set has codimension 2. Popov gives an example where this (and regularity on X) fails. A further example when $A\mathscr{S}$ is not open derives from Example 3 of 11.4.

7.4. Assume from now on that X is a vector space on which A acts linearly.

Under the above assumption $R[X]$ is a polynomial algebra, so admits an A stable gradation given by degree.

Assume (in this subsection) that X admits a slice $\mathscr{S}$ in the sense of 7.3.

Since D, and hence D_1, is A stable, so is $dR[X]$. In particular for all $a \in A$ one has $ad \in dR[X]$, so d divides ad. By degree ad must be a scalar multiple of d. Thus d must be a semi-invariant for the action of A on X.

Assume that the action of A on X admits no proper semi-invariants. In this case $d \in R[X]^A$. Moreover we may view d as an element of $R[\mathscr{S}_{reg}]$ by restriction. Then Eqs. (1) and (4) of 7.3 combine to give

$$R[\mathscr{S}_{reg}] \xleftarrow{\sim} \begin{cases} R[X]^A: & \operatorname{codim}_X D \geqslant 2, \\ R[X]^A[d^{-1}]: & \operatorname{codim}_X D = 1. \end{cases} \tag{5}$$

As announced in 1.2, we shall say that a slice $\mathscr{S}$ is affine if its closure $\overline{\mathscr{S}}$ takes the form $\overline{\mathscr{S}} = y + V$, for some $y \in X$ and some vector subspace V of X, necessarily of dimension $\iota_{X,A}$. Notice that this means that $\mathscr{S}_{reg}$ is open in $y + V$. Its complement D'' is closed and if $\operatorname{codim}_{y+V} D'' = 1$ we define d'' similarly to d in 7.3. Identifying $R[y + V]$ with the polynomial algebra $S(V^*)$ we obtain as in Eq. (2) of 7.3

$$R[\mathscr{S}_{reg}] \xleftarrow{\sim} \begin{cases} S(V^*): & \operatorname{codim}_{y+V} D'' \geqslant 2, \\ S(V^*)[d''^{-1}]: & \operatorname{codim}_{y+V} D'' = 1. \end{cases} \tag{6}$$

On the other hand restriction of functions gives an embedding $R[X]^A \hookrightarrow R[y+V] \xrightarrow{\sim} S(V^*)$ which through Eqs. (5) and (6), as they are also defined by restriction of functions, induces an isomorphism $R[X]^A[d^{-1}] \xrightarrow{\sim} S(V^*)[d''^*]$. Thus $d^{-1} \in S(V^*)[d''^{-1}]$ and conversely $d''^{-1} \in S(V^*)[d^{-1}]$, since $R[X]^A \subset S(V^*)$. Yet $S(V^*)$ is polynomial hence factorial having only scalars as invertible elements. Thus we conclude that d and d'' are proportional. (This was a little surprising – at least it surprised Vinberg.) In the above notation and hypotheses this gives the following

Proposition. *Suppose $R[X]$ admits no proper semi-invariants. Then* $\operatorname{codim}_{y+V}((y + V) - \mathscr{S}_{reg}) \geqslant 2$ *if and only if* $\operatorname{codim}_X(X - A\mathscr{S}_{reg}) \geqslant 2$. *If either hold $R[X]^A \xrightarrow{\sim} S(V^*)$.*

Remark. Unfortunately we cannot say too much in the case that d (or d'') is not scalar. In 11.4 we give an example even when $D = D''$; but $R[X]^A$ is not polynomial.

7.5. Since $R[X]$ is a polynomial algebra it admits unique factorization. Recall that the action of A preserves degree.

Through the above an argument (of Chevalley/Dixmier) shows that a semi-invariant of $\operatorname{Fract} R[X]$ is the ratio of semi-invariants of $R[X]$. (Indeed write $x^{-1}y \in \operatorname{Fract} R[X]$ with $x, y \in R[X]$ coprime. Then the semi-invariance of $x^{-1}y$ implies by unique factorization that x divides ax, for any $a \in A$. By degree ax must be a scalar multiple of x, so x must be a semi-invariant. The same holds for y.)

Now assume that $R[X]$ admits no proper semi-invariants. Then by the above,

$$\left(\operatorname{Fract} R[X]\right)^A = \operatorname{Fract} R[X]^A. \tag{4}$$

Suppose now that $\mathbf{k}$ has characteristic zero. Then by Rosenlicht's theorem [35] one has trdeg $\operatorname{Fract} R[X]^A = \iota_{X,A}$. (It is my understanding that Rosenlicht developed his result in studying a book of Chevalley in which this result was described. The precise result we require is Lemme 7 in Dixmier's paper [8] in which the key step is due to Chevalley appearing in the book "Géometrie Algèbrique" by C. Chevalley which is stated as being in preparation.)

Now let $\mathscr{S}$ be an affine slice (open dense in some $y + V$). Then by Eqs. (5), (6) of 7.4 and the above, we obtain $\dim V = \iota_{X,A}$. We remark that $A(y + V)$ which contains the open dense subset $A\mathscr{S}_{reg}$ need not be open in X, lie entirely in X_{reg}, nor contain the latter – see the examples in 11.4.

7.6. Assume that there exists an affine translate $y + V$ of a vector subspace V of X such that $(y + V)_{reg} := (y + V) \cap X_{reg}$ is non-empty, hence open dense in $y + V$ and such that restriction map $R[X]^A \to R[y + V]$ induces an isomorphism of rings of fractions. Then we shall say that $y + V$ is a rational slice to the action of A on X. In this case the (finitely many) generators of $R[y + V]$ (for example a basis of V^*) viewed as images of the restriction map can be taken to have a common denominator d. Let D be the locus of zeros of d. Let D be the zero locus of d. Then the invariant functions on X separate any two points of $(y + V) - D$.

As announced in 1.3 we shall say that $y + V$ is an algebraic slice if it is a rational slice, so in particular $(y + V) \cap X_{reg}$ is non-empty and the restriction map $R[X]^A \to R[y + V]$ is (already) an isomorphism, equivalently $d = 1$ in the above.

We remark that an algebraic slice is what Popov [34, 2.2] calls a Weierstrass section at least for the case when A is reductive. Popov calls a rational section [34, 1.1.1,1.2] a subvariety Y of X such that restriction of functions gives on isomorphism of $R(X)^A$ onto $R(Y)$. In [34, Example (2.2.9)] with A semisimple and $R[X]^A$ polynomial, Popov gives an example which neither admits a Weierstrass section, nor (for the same reason) a rational section. However in this case X is not the coadjoint module. A theorem of Rosenlicht [36] asserts if A is a connected solvable linear algebraic group acting regularly on an affine variety X then there exists a subvariety Y of X such that restriction of functions gives an isomorphism of $R(X)^A$ onto $R(Y)$, that is to say the action of A on X admits a rational section in the sense of Popov. Moreover if X is a vector space on which A acts linearly then "then it seems linearity of sections may be furnished", that is to say Y can be assumed to be a rational slice in our sense. The above quotation was taken from a letter to me from V.L. Popov. For the moment I have no further confirmation and particularly in view of the question raised in 7.11 it would be worthwhile to check this out.

Assume that $y + V$ is an algebraic slice. Then the invariant functions separate the points of $y + V$. In particular each A orbit through $A(y + V)$ meets $y + V$ at exactly one point. On the other hand $R[X]^A$ could be reduced to scalars and we could take $V = 0$. Then $y + V$ is reduced to just the (regular) element y.

7.7. Assume from now on that the base field $\mathbf{k}$ is algebraically closed of characteristic zero. Set Lie A$=\mathfrak{a}$. One has $T_{x,Ax}=\mathfrak{a}.x$, for all $x\in X$.

Assume that $R[X]$ admits no proper semi-invariants. Then by 7.5 if $y+V$ is a rational slice we must have $\dim V=\iota_{X,A}$. Since $y+V$ is dense in $(y+V)_{reg}$ it follows that $A(y+V)$ is dense in X.

Set $\mathscr{S}=\{s\in y+V \mid T_{s,As}\cap T_{s,y+V}=\mathfrak{a}.s\cap V=0\}$, which is open in $y+V$. Similarly $O:=\{s\in y+V \mid T_{s,As}+T_{s,y+V}=T_{s,X}\}$ is open in $y+V$.

Since $A(y+V)$ is dense in X, it follows that O is a non-empty subset of $y+V$.

Since $\dim V=\iota_{X,A}$, it follows that $O=\mathscr{S}\cap(y+V)_{reg}=\mathscr{S}_{reg}$. By construction the A orbits meeting $\mathscr{S}$ cut transversely at each point.

We conclude that if $y+V$ is a rational slice, then with D and $\mathscr{S}$ defined as above $\mathscr{S}\cap((y+V)-D)$ is an affine slice.

Conversely suppose that $\mathscr{S}$ is an affine slice, so then $\mathscr{S}_{reg}$ is open (dense) in some affine translate $y+V$ of a vector subspace of X. Then by Eqs. (5), (6) of 7.4 we conclude that $y+V$ is a rational slice.

The above may be summarized as follows.

Lemma. *Suppose $R[X]$ has no proper semi-invariants. Then the geometric notion of an affine slice is equivalent to the algebraic notion of a rational slice.*

7.8. Assume that $R[X]$ admits no proper A semi-invariants.

Let $y+V\subset X$ be a translate of a vector subspace V of X such that every orbit in $A(y+V)$ meets $y+V$ at just one point.

Now suppose in addition there exists a family $\chi(t)$: $t\in\mathbf{k}^*$ of linear automorphisms of X belonging to the normalizer of A in $GL(X)$ such that for all $t\in\mathbf{k}$

(1) $\chi(t)$ belongs to the normalizer of A in $GL(X)$, and so induces an element of $\operatorname{Aut}\mathfrak{a}$.
(2) $\chi(t)y=t^{-1}y$.
(3) $\chi(t)v_i=t^{m_i}v_i$, for some basis v_i of V and non-negative integers m_i.

By linearity and (1), $\dim At\chi(t)z$ is independent of $t\in\mathbf{k}$. By (2) and (3) one has $t\chi(t)(y+V)\subset y+V$, for all $t\in\mathbf{k}$.

The above allows one to make the following “deformation” construction. Fix $z\in y+V$ and set $\mathscr{M}:=\{t,t\chi(t)z)$: $t\in\mathbf{k}^*$, which is curve in $\mathbf{k}\times y+V$. Let ξ_y, ξ_i be the dual basis to the given basis on $\mathbf{k}y\oplus V$ and ξ the projection onto the first factor. Let p denote the projection onto the second factor. Since $\mathbf{k}$ is infinite, there is no polynomial which vanishes on $\{t\}_{t\in\mathbf{k}^*}$. It follows that the equations defining this curve are just $\xi_y=1$, $\xi_i=c_i\xi^{m_i+1}$ given $z=y+\sum_i c_iv_i$. In particular $y\in p(\overline{\mathscr{M}})$.

Proposition.

(i) *For all $z\in y+V$, one has a direct sum $T_{z,Az}\oplus T_{z,y+V}=T_{z,X}$. Moreover* $\dim V=\iota_{X,A}$.
(ii) *$y+V\subset X_{reg}$ and $\mathscr{S}:=y+V$ is a affine slice.*
(iii) $R[X]^A\xrightarrow{\sim}S(V^*)$.
(iv) $\operatorname{codim}_X(X-A(y+V))\geqslant 2$.

Proof. One has $T_{z,Az} = \mathfrak{a}.z$ and $T_{z,(y+V)} = V$. In particular the conclusion of (i) holds for $z = y$ by the hypotheses of the proposition. Moreover $\dim V = \operatorname{codim}_X T_{y.Ay} \geqslant \iota_{X,A}$.

Fix $z \in y + V$ and define $\mathscr{M}$ as above. Set $z_t = t\chi(t)z \in p(\mathscr{M})$. For all $t \in \mathbf{k}^*$ one has $T_{z_t,Az_t} = [\mathfrak{a}, z_t] = t\chi(t)[\mathfrak{a}, z] = t\chi(t)T_{z,Az}$, whilst $T_{z_t,y+V} = V = t\chi(t)V$. Thus the values of $\dim T_{z',Az'}$ and $\dim(T_{z',Az'} + T_{z',(y+V)})$: $z' \in p(\overline{\mathscr{M}})$ are constant on $p(\mathscr{M})$ and take their minimal value on its boundary point y. In particular for all $z \in y + V$, the first paragraph implies that the dimension of $T_{z,Tz} + T_{z,y+V}$ is at least that of $T_{z,X}$ implying equality of these spaces.

We conclude from the above that the morphism $A \times (y + V) \to X$ is smooth, so has an open (hence dense) image. Thus $(y + V)_{reg}$ is non-empty, hence open dense in $y + V$.

Again $\{z \in y + V \mid T_{z,Az} \cap T_{z,y+V} = \mathfrak{a}.z \cap V = 0\}$, is open in $y + V$, hence open dense, since it contains $\{y\}$.

Since the above two subsets of $y + V$ must intersect we conclude that $\dim V \leqslant \iota_{X,A}$. By our previous inequalities this forces $\iota_{X,A} \leqslant \operatorname{codim} T_{z,Az} \leqslant \operatorname{codim} T_{y,Ay} \leqslant \iota_{X,A}$ for all $z \in y + V$. Reinsertion in the above gives (i).

The first part of (ii) follows from (i). Already in the proof of (i) we noted that $\overline{A(y + V)} = X$. Moreover by (i) each A orbits cuts $y + V$ transversally. Hence (ii). Then (iii) and (iv) follow from 7.4. □

7.9. Consider the case of coadjoint action, that is $X = \mathfrak{a}^*$, where $\mathfrak{a} = \operatorname{Lie} A$.

Assume that $S(\mathfrak{a})$ to admit no proper semi-invariants.

Suppose that $\mathscr{S}$ is a locally closed subset of X such that restriction of functions induces an embedding of $S(\mathfrak{g})^A$ into $R[\mathscr{S}]$. Then $A\mathscr{S}$ is dense in X. Indeed otherwise the ideal I of definition of its closure is non-zero and A invariant. Then by a result of Dixmier, Duflo and Vergne [11], it contains a semi-invariant, so under the above hypothesis $I \cap S(\mathfrak{a})^A \neq \phi$, contradicting injectivity.

In particular for adjoint action we do not need to assume that $(y + V)_{reg}$ *is non-empty in the definition of an algebraic or rational slice.*

7.10. By contrast to the conclusion in 7.9 let X be the defining (n-dimensional) representation of $A = GL(n)$. Let $R[X]_+$ be the augmentation ideal of $R[X]$. It is A invariant. On the other hand $R[X]^A$ reduces to scalars and so $R[X]_+ \cap R[X]^A = 0$. Take $\mathscr{S} = \{0\}$, which is a single A orbit and the zero locus of $R[X]_+$. Thus restriction induces an isomorphism of $R[X]^A$ onto $R[\mathscr{S}]$. Yet $\mathscr{S}_{reg}$ is empty. (As is well known $X - \{0\}$ is a single A orbit.)

7.11. A problem of Dixmier [10, Prob. 4] is that the centre of the fraction field of an enveloping algebra is pure. This holds for $\mathfrak{a}$ completely solvable by a theorem of Bernat [10, Prop. 4.4.8]. The corresponding (also unresolved) question in the commutative set-up is whether $(\operatorname{Fract} S(\mathfrak{a}))^A$ is pure – irrespective of whether $\mathfrak{a}$ is algebraic. For $\mathfrak{a}$ algebraic it would be enough to show that coadjoint action admits an affine slice. We suggest that this might always be so.

More generally one can ask if $R(X)^A$ is always pure for A a Lie group acting linearly on a vector space X. Popov [34, 1.5.2] remarks that this is known as E. Noether's problem and that it has a negative answer even if A is a finite group. However if A is connected, then at least at that time no counter-examples were known. Again as noted in 7.6 even when $R(X)^A$ is pure a rational section need not exist. This example is for A semisimple and suggests, but does not necessarily imply, that the conjecture of the previous paragraph (which concerns the coadjoint representation) is false. On the other hand the remarks in 7.6 concerning a theorem of Rosenlicht indicates that the above suggestion does hold for $\mathfrak{a}$ solvable.

In this connection R. Tange remarked that (in the situation of the previous paragraph) $R(X)$ is not always pure over $R(X)^A$. Actually I had brought up the question with Dixmier about 25 years ago because of its interest to the Gel'fand–Kirillov conjecture. Serre informed Dixmier by letter that the answer was false even for $A = SL(2)$. Latter Alev, Ooms and Van den Bergh [1] showed that this indeed led to a counter-example to the Gel'fand–Kirillov conjecture. More recently it has been reported [7] that this purity question even fails for $\mathfrak{g}$ simple not of types A, C, G_2 for X the coadjoint module.

7.12. We digress slightly to discuss possible generalizations of the result of Dixmier, Duflo and Vergne mentioned in 7.9.

In the case of the enveloping algebra $U(\mathfrak{a})$ a result of Moeglin [29] asserts that any non-zero two sided ideal I of $U(\mathfrak{a})$ meets its semicentre $Sz(\mathfrak{a})$. This is already quite difficult and uses Duflo's characterization (see [10, Chap. 9]) of minimal primitive ideals. For the case of a simple Lie algebra $\mathfrak{g}$ an easy proof was pointed out to me by M. Gorelik which goes as follows. Since the common annihilator of all simple finite dimensional modules in $U(\mathfrak{g})$ is zero, there exists a simple finite dimensional module V such that the image of I in $\operatorname{End} V$ is non-zero and a $U(\mathfrak{g})$ bimodule. Yet by Jacobson's theorem the image of $U(\mathfrak{g})$ is the whole of $\operatorname{End} V$ and since the latter is a simple algebra the image of I must also be $\operatorname{End} V$ and so contains a copy of the identity. By complete reducibility we can take its inverse image in I to be $\mathfrak{g}$ invariant. This argument extends to the quantized enveloping algebra $U_q(\mathfrak{g})$. More generally let H be a Hopf algebra and Q an ad H stable (necessarily two-sided) ideal of H. One can ask if Q contains a one-dimensional H submodule. A related question (of Dixmier) is that of hearts. Let U be a prime quotient of an enveloping algebra $U(\mathfrak{a})$. Then does an ad $U(\mathfrak{a})$ stable ideal of U admit a one-dimensional submodule? This time the answer is no even for $\mathfrak{a}$ semisimple [4].

7.13. Assume in this section that $\mathfrak{a}$ is an algebraic Lie algebra with $Y(\mathfrak{a}) = Sy(\mathfrak{a})$, that is $S(\mathfrak{a})$ has no proper semi-invariants.

In previous work [21,22], we had constructed an algebraic slice for coadjoint action through an adapted pair defined to be a pair (h, y) with $h \in \mathfrak{a}$ semisimple, $y \in \mathfrak{a}^*_{reg}$ satisfying $(\operatorname{ad} h)y = -y$ and V an ad h stable complement to $(\operatorname{ad} \mathfrak{a})y$, with two extra conditions on V. First that the sum of the ad h eigenvalues on V counted with multiplicity equals $\frac{1}{2}(\dim \mathfrak{a} - \operatorname{index} \mathfrak{a})$. Second that these eigenvalues are all non-negative.

The latter condition is also that used in 7.8. It was shown to imply that $y + V \subset \mathfrak{a}^*_{reg}$ (still $A(y + V)$ need not be the whole of $\mathfrak{a}^*_{reg}$, [21, 8.5,8.7]) and that every A orbit through $y + V$ cuts transversally. In particular $y + V$ is an affine slice. Recalling 7.3, we may conclude that the morphism $A \times (y + V) \longrightarrow X$ defined by the action is smooth and hence $A(y + V)$ is open (dense) in X. This last conclusion may fail for an arbitrary algebraic slice – see Examples 2, 3 in 11.4.

Notice that 7.8(iv) implies that $\operatorname{codim}_{\mathfrak{a}^*}(\mathfrak{a}^* - A(y + V)) \geqslant 2$. This is better than what we had known previously for an adapted pair, namely that $\operatorname{codim}_{\mathfrak{a}^*}(\mathfrak{a}^* - \mathfrak{a}^*_{reg}) \geqslant 2$ – see 1.11.

We take this opportunity to remark that the first condition on the eigenvalue sum in V is automatic via a result of Dixmier, Duflo and Vergne [11] and a remark of my former student Doron Shafrir. Again if $Y(\mathfrak{a})$ is polynomial the second condition is also automatically satisfied even under a slightly weaker condition than there being no non-trivial semi-invariants. Otherwise it may fail (under this slightly weaker condition). Details are given in [25].

As we shall see Theorem 9.3, in the present work we stumbled upon the fact that in most cases (outside types A, C, B_{2m}, F_4) the truncated Borel $\mathfrak{b}_E$ admits an algebraic slice $y + V$ *without* y being regular *nor* $h \in \mathfrak{b}_E$. Previously we had not seen how it could be possible to achieve this.

7.14. Let us show how the above analysis may be used to recover the Cayley–Hamilton theorem. The classical proof of the latter known to the author (though there may be several) goes as follows. We adopt the notation of 7.3.

Recall the notation of 7.1. Let $\xi_{i,j}$ be the dual basis to $x_{i,j}$ and $R[M]^G$ the algebra of G invariant polynomials on M.

For $i = 1, 2, \dots, n$, view $\bigwedge^i E$ as a G module through diagonal action and set $X_i = g \mapsto tr(g, \bigwedge^i E)$: $g \in G$. Alternatively one may define the $X_i(g)$ as the coefficients in the characteristic polynomial of g.

As is well known and easy to check X_i: $i = 1, 2, \dots, n$ is a G invariant homogeneous polynomial on M of degree i, whose restriction to the space $C = x + V$ of companion matrices is up to a non-zero scalar $\xi_{n,n-i+1}$, and of course these elements generate $R[C]$.

On the other hand it is clear that the set $D = M - \Omega$ of matrices not admitting a cyclic vector form a proper closed subset of M. Thus $\Omega = GC$ is open dense in M. Consequently the restriction to C induces an injection (and hence bijection) of $R[M]^G$ onto $R[C]$. We call it the Cayley–Hamilton bijection. All this needs no particular assumption on $\mathbf{k}$.

We now establish the Cayley–Hamilton bijection without the explicit construction of generators for $R[M]^G$. However we shall assume $\mathbf{k}$ algebraically closed of characteristic zero.

First we recall some easy to prove well-known facts.

The trace form $\langle a, b\rangle \mapsto tr(ab)$: $a, b \in M$ is a non-degenerate invariant bilinear form on M.

Take $y \in M$ as in 7.1. The subalgebra M^y of matrices commuting with y is generated by y and in particular has dimension n, which in turn is the minimal codimension of a G orbit in M.

Take $V \subset M$ as in 7.1. The trace form restricts to a non-degenerate pairing $M^y \times V \to \mathbf{k}$, whilst $\langle M^y, [M, y]\rangle = 0$. Thus $M = V + [M, y]$, and the sum is direct.

There is a unique trace-zero diagonal matrix h satisfying $[h, y] = -y$. Moreover $[h, V] \subset V$ and the eigenvalues for this action are non-negative.

The conjugation action of G on M admits no proper semi-invariants.

As shown in 7.1, every G orbit in Ω meets C at just one point.

From the above the hypotheses of 7.8 are met and the Cayley–Hamilton bijection results form its conclusion. Notice that we may also conclude that D has codimension $\geqslant 2$ in M.

We remark that a well-known but more complicated argument shows that D has codimension 3 in $M(n)$. Indeed by say [21, 8.7] one has $D = M(n) - M(n)_{reg}$. In the language of Dixmier [9], M is a disjoint union of sheets with Ω being the regular sheet. Apparently only the regular sheet admits the remarkable "linearization" described in 7.1. (For more on Dixmier sheets – see [2,3].)

Through the existence of a principal s-triple which provides a three-dimensional subspace S with $S_{reg} = S - \{0\}$, it follows that $M - M_{reg}$ has codimension at least three and it must be exactly three since this is the codimension of the subregular sheet. Instead this latter estimate may now be obtained from [31, Prop. 1.6(2)].

In view of 12.4 we may conclude that faithfully flat descent gives a proof of the Cayley–Hamilton theorem, free from the construction of invariants. It would be pleasing to attack the conjecture in 7.11 by this approach.

The main obstacle to extending our proof of the Cayley–Hamilton theorem is that one certainly needs the base field to be infinite for the proof of Proposition 7.8. In the present case one could probably just check its conclusion by explicit computation. However there is hardly much point since besides we need its conclusion elsewhere.

8. Switching

8.1. The metaslice described in 6.2 is not quite the result we promised in 1.10. The question then arises as to whether we may recover the desired result by a process of "switching" the positive and negative Borel subalgebras. One may further ask if it is thereby possible to morph a metaslice into an algebraic slice in the sense of 7.4.

8.2. Recall the notation of 1.6, 1.5. Whilst $Y(\mathfrak{b}_E)$ is always polynomial, there may not be an algebraic slice to the coadjoint action. Indeed as noted in 11.4 this is already the case in type C_2. On the other hand the conclusion of Theorem 6.2 captures the essential *algebraic* feature of an algebraic slice in that it linearizes the generators of the invariant algebra, without necessarily having its geometric interpretation.

8.3. Suppose c is odd, so then $\mathfrak{g}$ is of type A_{2m}. Since $sg(i) = -1$ for all $i \in I_a$, it follows that $A(\mathfrak{g})^{\varsigma}$ is generated by the subspaces $Y(\mathfrak{b}_E^-)_{-\mu_i} Y(\mathfrak{n})_{\mu_i}: i \in I_a$. Moreover the restriction to $y + V$ described in 6.2, sends by 3.6, each one-dimensional subspace $Y(\mathfrak{n})_{\mu_i}$ to scalars. Consequently in this case Theorem 6.2 exhibits $y + V$ as an algebraic slice for coadjoint action of $\mathfrak{b}_E$. In particular it recovers in a little more detail a special case of the result proved in [22] for all biparabolics in type A.

8.4. The above circumstance raises the question as to whether by modifying the affine subspace $y + V$, we can switch signs so that we may recover an algebraic slice from Theorem 6.2 in all cases. However we already know this to be impossible (see 8.2). Yet here we shall prove that it is possible to switch all signs which are associated to elements of I which are not κ fixed. This will achieve two goals. First it will recover an algebraic slice in type A_{2m+1}, secondly it will enable us to present the main result (Theorem 6.2) in a more congenial form, although besides type A only type E_6 is affected.

8.5. Define I_0 as in 5.1. We can assume I_0 not empty. This means that $\mathfrak{g}$ is of one of the following types A, D_m: m odd, E_6, in particular $\mathfrak{g}$ is simply-laced. We can and do assume I_0 chosen so the corresponding simple reflections commute pairwise.

Lemma. *Fix $i \in I$. There exists a unique $\gamma \in \Pi_{\tau^{op}(i)}$ such that $\alpha_i^\vee(\gamma)\gamma^\vee(\varpi_i) \neq 0$. Moreover $\alpha_i^\vee(|\gamma|) = |\gamma|^\vee(\varpi_i) = 1$.*

Proof. The passage from relation (1) of 3.3 to relation (5) of 3.4, similarly gives a further three relations starting from (2)–(4) of 3.3. In the notation of 5.4, these may be summarized by the one relation for c even.

$$\varpi_i + \varpi_{\kappa(i)} = \sum_{\gamma \in \Pi_{\tau^{op}(i)}} \gamma^\vee\big(\kappa(\varpi_i)\big)\gamma. \tag{1}$$

Via Eq. (7) of 3.6, this also holds for c odd.

By the pairwise orthogonality of the elements of $\Pi_{\tau^{op}(i)}$, the above relation implies that $\gamma^\vee(\kappa(\varpi_i)) = \gamma^\vee(\varpi_i)$, even if γ is not κ stable. Evaluate both sides of (1) at $\alpha_i^\vee$ taking account of this last remark. Since $\kappa(i) \neq i$, obtain

$$1 = \sum_{\gamma \in \Pi_{\tau^{op}(j)}} \gamma^\vee(\varpi_i)\alpha_i^\vee(\gamma).$$

It remains to show that no summand in the right-hand side can be strictly negative. Otherwise $\alpha_i + |\gamma|$ is a positive root whose coefficient in α_i is $\geqslant 2$. This is only possible in type E_6 with $i \in I_b$. Then $\tau^{op}(i) = a$. This case can be easily excluded, for example by examining Π_a for type E_6 given in 11.3. $\square$

8.6. Fix $i \in I_0$ as above. Let γ denote the unique element of $\Pi_{\tau^{op}(i)}$, in the conclusion of Lemma 8.5. Set

$$\theta'(x_i) = s_i|\gamma| + \alpha_i = |\gamma|. \tag{$*$}$$

By Corollary 2.16 and (ii) of Lemma 4.9 one has

$$h_*\big(\theta'(\alpha_i)\big) = 1.$$

This is used below in computing values of generators but we wont mention it again as we are just carrying out calculations similar to those made in Sections 3, 5, 6.

Recall X_i defined in 6.2. The choice of ς_i was made to ensure that restriction to $y + V$ sends the first factor in X_i to scalars and the second factor to linear functions.

We call an i-switch, a modification $y' + V'$ of $y + V$ such that restriction of X_i to $y' + V'$ sends the first factor in X_i to linear functions and the second factor to scalars, but with no change with respect to the remaining X_j: $j \in I - \{i, \kappa(i)\}$.

In the above we may assume that I is not just reduced to a single κ orbit, since this is just type A_2 and we can perform the unique i-switch through ς. By 2.21, this is the only case when Π meets either π or $-\pi$. Thus no element of Π can change sign under a simple reflection (or indeed under a product of commuting simple reflections). This is used below in computing values of generators at y', but we wont bother to mention it again.

To construct an i-switch set $y' = s_i y$. The definition of V' is a little more complex. Set $V_i = \mathbb{C}x_{\alpha_i} + \mathbb{C}x_{\theta(\alpha_i)}$. Let V_i' denote the corresponding expression when θ' replaces θ.

By our previous construction and 5.4, 6.2 either V_i or $\varsigma(V_i)$ is a subspace of V, depending on those pesky signs again! Assume the former. Then V_i admits a unique complement V^i in V spanned by root vectors. Set $V' = s_i(V^i) \oplus \varsigma(V_i')$. If the latter holds insert the obviously needed factors of ς.

Proposition *(c even). Suppose $i \in I_0$. Then $y' + V'$ is an i-switch.*

Proof. Recall Eq. (6) of 4.3 and its analogue $(*)$ above which we repeat below to aid comparison.

$$\varpi_i + \varpi_{\kappa(i)} = \alpha_i + \sum_{\gamma \in \Pi_{\tau(i)}} \gamma^\vee\big(\kappa(\varpi_i)\big)\gamma, \tag{1}$$

$$\varpi_i + \varpi_{\kappa(i)} = \sum_{\gamma \in \Pi_{\tau^{op}(i)}} \gamma^\vee\big(\kappa(\varpi_i)\big)\gamma. \tag{2}$$

Apply s_i to both sides of (1) and (2). Noting that $\alpha_i \neq \kappa(\alpha_i)$, we obtain

$$\varpi_i + \varpi_{\kappa(i)} = \sum_{\gamma \in s_i \Pi_{\tau(\alpha_i)}} \gamma^\vee\big(\kappa(\varpi_i)\big)\gamma, \tag{3}$$

$$\varpi_i + \varpi_{\kappa(i)} = \alpha_i + \sum_{\gamma \in s_i \Pi_{\tau^{op}(\alpha_i)}} \gamma^\vee\big(\kappa(\varpi_i)\big)\gamma. \tag{4}$$

Note that there is just an interchange as we pass from the pair (1), (2) to the pair (3), (4), *except* with respect to the domain of summation. It is this interchange which effects the switching.

Now we consider the matrix coefficients $b_{(j)}, b''_{(j)}, b^{op}_{(j)}, b''^{op}_{(j)}$, for $j \in I/\langle\kappa\rangle$ described in 3.4, 3.6, 5.4 and 6.1. We show that they are non-vanishing at suitable values and observe that for $j = i$ there is an interchange as to whether they lead to scalar values or linear functions, whilst for $j \notin \langle\kappa\rangle i$, there is no change. All this is pretty trivial but we fill in the details anyway.

Suppose first that $j = i$.

As previously we write

$$a_i = gr_{\mathscr{F}} b_{(i)}, \qquad a_i^{op} = gr_{\mathscr{F}} b_{(i)}^{op}.$$

Recall that in 3.4 we had used (2) to show that $a_i(y)$ is a non-zero scalar, the key points being orthogonality and degree comparison. Similarly we may use (3) to show that $a_i^{op}(y')$ is a non-zero scalar.

Now we show that $a_i(y' + \mathbb{C}x_{\theta'(\alpha_i)}) = \varsigma(\mathbb{C}x_{\theta'(\alpha_i)})$, using (4), above. Here we have ignored an overall sign and we remark that the presence of ς just arises from the duality resulting from the Killing form.

In principle the above is quite tricky because the summation set has a slightly different nature to what it had before. (In particular it is not κ stable.)

Actually the required result is almost immediate. Indeed by Lemma 8.5, the only term in the summation of (4) affected by s_i is just γ in the conclusion of Lemma 8.5. Moreover absorbing α_i in $s_i|\gamma|$, using $(*)$ of 8.6 and comparison with (2) shows that the non-vanishing of y on a_i implies its non-vanishing on $y' + \mathbb{C}x_{\theta'(\alpha_i)}$, for all non-zero values of the scalars in $\mathbb{C}$. This produces the required linear function. (One must also check that no other terms occur from the second evaluation. However this is just a consequence of the linear independence of the elements of $\{\alpha_i, \Pi_{\tau^{op}(i)}\}$, which implies the uniqueness of their sum equaling $-\operatorname{wt} a_i$.)

Notice in the above that orthogonality holds – see 5.6. (This is a surprise at least for type E_6.) It means that we may show that the evaluation of $a''_i := gr_{\mathscr{F}} b''_{(i)}$ on $(y' + \mathbb{C}x_{\alpha_i})$ is just $\varsigma(\mathbb{C}x_{\alpha_i})$, by the trick used in the last part of the proof of 6.1.

Finally suppose $j \notin \langle\kappa\rangle i$. Then the invariants a_j and a''_j (the latter defined exactly when j is not κ fixed) are $\langle s_i, s_{\kappa(i)}\rangle$ invariant. In view of our choice of y', V', the required assertion is trivial.

This completes the proof of the proposition. $\square$

Remark. Recall that I_0 is chosen so that the $s_i \in I_0$ commute pairwise. Let I'_0 be any subset of I_0 and set $s = \prod_{i \in I'_0} s_i$. We may carry out successively the i-switches for all $i \in T'_0$ and this in any order. In this y is replaced $y' := sy$ and V by V' defined by replacing the summand $\bigoplus_{i \in I'_0} V_i$ by $\bigoplus_{i \in I'_0} s_i V'_i$ and applying s to the resulting expression.

8.7. A result similar to Proposition 8.6 may be obtained for c odd. Recall Eq. (1) of 4.12 and the notation used there. Again from Eq. (7) of 3.6 we obtain (as in 3.4) that

$$\varpi_\alpha + \varpi_{\kappa(\alpha)} = \sum_{\gamma \in \Pi_b} \gamma^\vee(\varpi_i)\gamma, \quad \forall \alpha \in \pi_a. \tag{1}$$

Now assume that α is not a central root (see 4.12 for terminology), so then $\kappa(\alpha)^\vee(\alpha) = 0$. In this case one checks that $\alpha_c^\vee(\varpi_\alpha + \varpi_{\kappa(\alpha)}) = 1$. Thus applying the reflection s_{α_c} to Eq. (1) above we obtain Eq. (1) of 4.12 and vice-versa, except (as before) for the change in summation index. We further observe that since $h_*(\alpha_c) = 0$ (because α is not central), s_{α_c} cannot change the sign of any $\gamma \in \Pi$ (because $h_*(|\gamma| = 1)$) nor can it change the value of h_*. Further define V_i, V^i, V_i' as above. Then exactly as in 8.6 we obtain the

Proposition *(c odd). Assume that α_i is not a central root. Set $y' = s_{(\alpha_i)_c} y$, $V' = s_{(\alpha_i)_c} V^i + \varsigma(V_i')$. Then $y' + V'$ is an i-switch.*

Remark. One has $((\alpha_i)_c, (\alpha_j)_c) = (\alpha_i, \alpha_j)$. Thus we may successively apply i-switches for all $i \in I_0 = I_a$ excluding the central root. To make up for the loss of the central root one may simply use the Chevalley antiautomorphism ς.

8.8. By Proposition 8.5 we may recast Theorem 6.2 into the following more congenial form. Set

$$A(\mathfrak{g}) = \sum_{\mu \in \Lambda} Y\bigl(n^-\bigr)_{-\mu} Sy(\mathfrak{b})_\mu.$$

For c even, note that $sg(i) = 1$ exactly when $i \in I_a$ and set $s = \prod_{i \in I_0 \cap I_a} s_i$. Set $y' = s$ and define V' as in Remark 8.6 with $I_0' = I_0 \cap I_a$. For c odd, take simply $y' + V' = y + V$.

Theorem. *Restriction of functions induces an isomorphism of $A(\mathfrak{g})$ onto $y' + V'$.*

Remark. This constructs the required metaslice in the sense of 1.10.

8.9. Generalized exponents. A natural question one may ask is whether our metaslice $y + V$ is an algebraic slice for G action. Since y is already regular nilpotent this holds if and only if V is a complement to $(\operatorname{ad}\mathfrak{g})y$. Just for the moment (that is in this subsection) let us rescale x, h, y so that the relations in our principal s-triple become $[h, x] = x$, $[h, y] = -y$, $[x, y] = h$. Then V is an ad h stable complement to ad $(\mathfrak{g})y$ if and only if the eigenvalues of ad$h$ on V are the so-called generalized exponents. The latter are just the eigenvalues of ad$h$ on the centralizer $\mathfrak{g}^x$ of x in $\mathfrak{g}$. Now each generalized exponent is the degree minus one of some homogeneous generator of $Y(\mathfrak{g})$, whilst the sum of the degrees is exactly $\frac{1}{2}(\dim\mathfrak{g} + \operatorname{index}\mathfrak{g})$, a rule now known to hold in remarkable generality [31, Prop. 1.4]. Since $\operatorname{index}\mathfrak{g} = \operatorname{rank}\mathfrak{g}$, the sum of the generalized exponents is exactly $\frac{1}{2}(\dim\mathfrak{g} - \operatorname{rank}\mathfrak{g})$.

On the other hand in [14, Sect. 6] we verified the mysterious fact that the sum of the degrees of the generators of $A(\mathfrak{g})$ can equal $\frac{1}{2}(\dim\mathfrak{g} + \operatorname{rank}\mathfrak{g})$ if and only if one mistakenly takes the ε_i: $i \in I$ to all equal 1. Consequently it is always less than or equal to this sum with equality if and only if $\varepsilon_i = 1$, for all $i \in I$, that is in types A and C. Coupled to this we showed in [13, 4.9], that when equality holds, then the degrees of both sets of generators individually coincide. (Actually in these case we showed that the generators of $Y(\mathfrak{g})$ may be constructed from the generators of $A(\mathfrak{g})$.) All this gives the following

Proposition. *The affine translate $y+V$ is an algebraic slice for the G action in $\mathfrak{g}^*$ if and only if $\mathfrak{g}$ is of type A or C.*

Remark. Thus $G(y+V)$ is dense in $\mathfrak{g}^*$ if and only if $\mathfrak{g}$ is of type A or C. Outside these types, the restriction of $Y(\mathfrak{g})$ to $y+V$ is *not* injective. Now $GA(\mathfrak{g})$ embeds in $R[G(y+V)]$ via restriction. Had we also known this for $(\operatorname{ad} U(\mathfrak{g}))A(\mathfrak{g}) = \mathbb{C}\,GA(\mathfrak{g})$, then we would be able to conclude that $(\operatorname{ad} U(\mathfrak{g})A(\mathfrak{g}))^G \subsetneq Y(\mathfrak{g})$ outside types A and C. This is because in these cases $\overline{G(y+V)} \subsetneq \mathfrak{g}^*$ and so its ideal of definition must contain a non-zero semi-invariant (7.9).

9. Exotic switching

9.1. The example of type G_2 in 11.4 indicates that it is possible to make further switchings associated to simple reflections defined by those $\alpha \in \pi$ on which h_* takes the value -1. Combined with results in Section 8, this means that nearly all switchings are possible and in most cases we will obtain an algebraic slice $y+V$ which lies in $\mathfrak{b}_E^*$ but not entirely in $(\mathfrak{b}_E^*)_{reg}$. In particular y is not regular and so it will not be possible to obtain such examples by our previous method of constructing an adapted pair. Worst still we cannot construct V as a complement to $(\operatorname{ad}\mathfrak{b}_E)y$ in $\mathfrak{b}_E^*$.

9.2. Set $\pi_{-1} := \{\alpha \in \pi \mid h_*(\alpha) = -1\}$. (Let I_{-1} denote the corresponding subset of the index set I.) Below we list some remarkable properties of this set. The proofs are case by case but very easy.

(1) π_{-1} is non-empty exactly in types B, D, E, F, G.

(2) $\pi_{-1} = \{\alpha \in \pi \,|\, \varepsilon_\alpha = 1/2\}$.

By (1) there is a unique simple root α_0 not orthogonal to the highest root β_*.

(3) If π_{-1} is non-empty, then $\alpha_0 \in \pi_{-1}$.

Complementary to (3), π_{-1} may be constructed inductively as follows. Let π_* be the set of simple roots of an indecomposable component of Δ_* not of type A or C. (There is always at most one.)

(4) One has $\pi_{-1} = \{\alpha_0 \cup (\pi_*)_{-1}\}$. Define a subset $\mathscr{K}_{-1}$ of the Kostant cascade $\mathscr{K}$ inductively as follows. Let the subset defined with respect to π_* be denoted by $(\mathscr{K}_*)_{-1}$ and set $\mathscr{K}_{-1} = \{\beta_*\} \cup (\mathscr{K}_*)_{-1}$. Let $\beta_{**\ldots}$ be the unique (by the comment just above (4)) smallest element of $\mathscr{K}_{-1}$.

It is always true that $\beta_* \in \Pi$ up to a (uniquely determined) sign (2.16, 2.20).

(5) Up to the above sign, $\mathscr{K}_{-1}$ belongs to the same subset of $\pm\Pi$ as does β_*. For example if $\beta_* \in -\Pi_a$, then $\mathscr{K}_{-1} \subset -\Pi_a$.

This can be checked from the tables for Π given in the Appendix, though possibly it could be proved inductively.

(6) The roots in π_{-1} (and in $\mathscr{K}_{-1}$) are orthogonal to those in $\{\alpha \in \pi \mid \kappa(\alpha) \neq \alpha\}$.

Both sets are non-empty only in types D_{2m+1}, E_6, so this assertion is rather easily checked.

As an ordered set, $\mathscr{K}$ is an inverted tree (see [18, Table III]), with β_* as its head. Then $\mathscr{K}_{-1}$ is a linearly ordered subset of $\mathscr{K}$ consisting of all elements between β_* and $\beta_{**\ldots}$.

Fix $\beta_i \in \mathscr{K}$. Then the subset $\mathscr{K}(i) := \{\beta_j > \beta_i\}$ is linearly ordered. Moreover (cf. [18, 4.12]), the weight of the corresponding generator a_i of $Y(\mathfrak{n})$ takes the form

(7) wt $a_i = \beta_i + \sum_{j\in\mathscr{K}(i)} n_j\beta_j$, with the n_j being non-negative integers since wt $a_i \in P^+$. (Actually the n_i are all positive though we do not need to know this.)

Take $\beta_i \in \mathscr{K}_{-1}$. Designate the corresponding element of π_{-1} as α_i, that is to say α_i is the unique simple root, not orthogonal to β_i, of the indecomposable root system for which β_i is the highest root. One has $\alpha_i \in \pi_{-1}$ and wt $a_i = \varpi_i$.

9.3. By (1) above we can assume c even. The highest root β_* belongs to one of the four subsets $\pm\Pi_a$, $\pm\Pi_b$. We assume that $\beta_* \in -\Pi_a$, the remaining three cases being exactly the same.

Fix $\beta_i \in \mathscr{K}_{-1}$ and take $\alpha_i \in \pi_{-1}$ as above. By 9.2 (5) and the above assumption we have $\beta_i \in -\Pi_a$.

Choose $y + V$ as in 6.2, that is to say choosing y by 3.4 $(*)$ and V by 5.1, 5.12.

By (2) above there is a generator a_i of $Y(\mathfrak{n})$ of weight ϖ_i. By our above assumption we obtain from Proposition 3.4 that $a_i(y) = a_i(y_a) \neq 0$. On the other hand by 5.4, the opposed element $\kappa(a_i)$ restricted to $y + V$, is just $x_{-\theta(\alpha_i)}$. Of course it is the evaluation at $y + \mathbb{C}x_{\theta(\alpha_i)}$ which produces this linear function. Let V_i denote the subspace of V in which the direct summand $\mathbb{C}x_{\theta(\alpha_i)}$ is omitted.

An exotic i-switch is implemented by replacing $y + V$ by $y' + V'$, where $y := s_i y$ and $V' := s_i(V_i) \oplus \mathbb{C}x_{-\beta_i}$. Notice that it is $x_{-s_i\beta_i}$ rather than $x_{-\beta_i}$ which appears in the summand which describes y', the latter having been "liberated" to serve in V'.

The remaining elements in the sum describing y which were needed to give $a_i(y)$ a scalar value all belong by (5), (7) above to $\mathscr{K}_{-1} \cap \mathscr{K}(i)$ and are therefore s_i invariant. Explicitly the (unique) monomial in a_i whose value on y is non-zero is, in the notation of (7), the expression

$$x_{\beta_i} \prod_{\beta_j \in \mathscr{K}_{-1}\cap\mathscr{K}(i)} x_{\beta_j}^{n_j}.$$

We conclude that the evaluation of a_i at $y' + V'$ is exactly the linear function x_{β_i}. On the other hand the evaluation of the opposed element $\kappa(a_i)$ at $y' + V'$ is not obviously a non-zero scalar. Because of this we simply omit it, replacing X_i of 6.2 by a_i. Finally as in the last part of 7.12 the evaluation of the remaining invariants is unaltered on replacing $y + V$ by $y' + V'$, since they are s_i invariant.

9.4. As a consequence of the roots in π_{-1} being pairwise orthogonal we may simultaneously carry out all exotic i-switches as follows. Set $s' = \prod_{i\in I_{-1}} s_i$ and $y' = s'y$. To be explicit we again assume that $\beta_* \in -\Pi_a$ and recall (5) above. Then $V = V_1 \oplus V_2$, $V_2 := \bigoplus_{i\in I_{-1}} x_{-\theta(\alpha_i)}$. Set $V' = s'(V_1 \oplus (\bigoplus_{i\in I_{-1}} s_i\mathbb{C}x_{-\beta_i}))$. Fix $i \in I_{-1}$. Then evaluation of the a_i on $y' + V'$, gives the linear function $\{x_{s's_i\beta_i}\}$, because a_i is $\langle s_j\colon j \in I_{-1} - \{i\}\rangle$ invariant. Similarly evaluation of the remaining invariants is unaltered as they are $\langle s_i\colon i \in I_{-1}\rangle$ invariant.

In view of (6) above we may also carry out the i-switches described in 8.5. This means further replacing the pair (y', V') above by y'', V'', obtained by the substitutions defined as 8.8. Notice that y'' is obtained from y by the product of the commuting reflections corresponding to each i-switch. At the same time we replace X_i by a_i, whenever $i \in I_{-1}$. We may also make this replacement, when $\kappa(i) \neq i$ and when $h_*(\alpha_i) = 1$. Restriction of functions to $y'' + V''$ then gives an exact analogue of Theorem 6.2 after carrying out these replacements in $A(\mathfrak{g})^{s\mathfrak{g}}$. In this extreme case when $\{i \in I \mid h_*(\alpha_i) = 0,\ \kappa(i) = i\}$ is the empty set (that is outside types C, B_{2n}, F_4) this algebra is simply $Sy(\mathfrak{b}) = Y(\mathfrak{b}_E)$. Write $y'' + V'' := y + V$, for c odd (see 8.8). Thus we have proved the following

Theorem. *Restriction of functions gives an isomorphism of $Y(\mathfrak{b}_E)$ onto $y'' + V''$, in types A, B_{2n+1}, D, E, G.*

Remark. Thus $y'' + V''$ is an algebraic slice and so admits an open dense subset $\mathscr{S}$ which is an affine slice (see 7.6). Outside type A only a dense subset of orbits in $B_E(y'' + V'')$ will be regular and generally only a dense subset of all regular orbits need be so obtained.

10. Affine slices

10.1. The construction of an affine slice in type C_2 given in 11.4 admits the following generalization.

Assume $-1 \in W$. This is equivalent to the condition that $V = \bigoplus_{\beta \in \mathscr{K}} \mathbb{C}x_{-\beta}$ has dimension equal to the rank of $\mathfrak{g}$ and hence to the number of generators of $Y(\mathfrak{b}_E) = Y(\mathfrak{n})$.

Let x be the product of the x_β as β runs over the non-minimal elements of $\mathscr{K}$ and let D be the zero locus of x in V. Set $\mathscr{S} = V - D$.

Lemma.

(i) $\mathscr{S} = V_{reg}$.
(ii) $T_{y,Ny} \oplus T_{y,V} = \mathfrak{n}^*$, $\forall y \in V$.
(iii) *$\mathscr{S}$ is an affine slice to coadjoint orbits in $\mathfrak{b}_E^*$.*

Proof. (i) is implicit in [18]. Indeed take $f \in V - D$. Through the decomposition of $\mathfrak{n}$ as a sum of Heisenberg Lie algebras given in [18, 2.2] it follows that $\ker f = \bigoplus_{\beta \in \mathscr{K}} \mathbb{C}x_\beta$, where one notes that the Heisenberg's associated to the minimal elements of $\mathscr{K}$ are all one-dimensional. Then apply [18, 2.4] which implies that index $\mathfrak{n} = |\mathscr{K}|$.

Since the elements of $\mathscr{K}$ are strongly orthogonal it follows that $(\operatorname{ad}\mathfrak{n})V \cap V = 0$. Thus $T_{y,Ny} \cap T_{y,V} = 0$, $\forall y \in V$. Since $\dim V = |\mathscr{K}|$, (ii) follows from (i).

For (iii), fix $\beta \in \mathscr{K}$. It is noted in [18, 2.8], that there exist non-negative integers $n_{\beta'}$ such that

$$\varpi(\beta) = \beta + \sum_{\beta' \in \mathscr{K} \mid \beta' < \beta} n_{\beta'}\beta',$$

is a dominant weight. Moreover by [18, 4.12] there exists a unique up to scalars element $a_{\varpi(\beta)} \in Y(\mathfrak{n}) = Sy(\mathfrak{b})$ of weight $\varpi(\beta)$ and in which the monomial

$$x_\beta \prod_{\beta' \in \mathscr{K} \mid \beta' < \beta} x_{\beta'}^{n_{\beta'}},$$

appears. (To be more precise the restriction of $a_{\varpi(\beta)}$ to V is a non-zero scalar multiple of this monomial.)

It follows that the restriction of $Y(\mathfrak{n})$ to V is the subalgebra of $R[V]$ generated by the above monomials. Dimensionality implies that this restriction is an embedding.

Starting from the highest root which is the unique maximal element of $\mathscr{K}$, we may inductively construct elements X_β in $Y(\mathfrak{n})[X_{\beta'}^{-1}\colon \beta' < \beta]$ whose restriction to $V - D$ is just x_β. Let X denote the product of the X_β as β runs over the non-minimal elements of $\mathscr{K}$.

It is clear from the above that the embedding of $Y(\mathfrak{n})$ into $R[V]$ induces an isomorphism of $Y(\mathfrak{n})[X^{-1}]$ into $R[V][X^{-1}]$. In particular the invariants in the former algebra separate the elements of $V - D$. Recalling that $Y(\mathfrak{n}) = Y(\mathfrak{b}_E)$, it follows that a B_E can pass at most once through $V - D$. In view of (ii) this establishes (iii). $\square$

10.2. The above result applies in particular to those truncated Borel subalgebras of $\mathfrak{g}$ simple for which Theorem 9.4 does not give an algebraic slice, namely in types C, B_{2n}, F_4. In type C it seems unlikely that we can do better. However in the remaining two cases a significant improvement is possible through exotic switching. Indeed we show that it suffices to localize at the highest root vector. In type B_{2n}, the calculation will also illustrate the implementation of exotic switching when several switches (in fact $n-1$) are made. We start with type F_4 in which only one exotic switch is carried out. We use the Bourbaki [5] labelling.

We say that $\Upsilon \subset \Delta$ defines an element y (resp. a subspace) of $\mathfrak{g}$ given $y = \sum_{\upsilon \in \Upsilon} x_{-\upsilon}$ (resp. $V = \bigoplus_{\upsilon \in \Upsilon} \mathbb{C}x_{-\upsilon}$). Define y'' by the subset $\{1342, 1121\}$ and V'' by the subset $\{2342, 1100, 0121, 1122\}$. Notice that 2342 is the highest root β_*, let D denote the zero locus of x_{β_*} in $y'' + V''$.

Lemma *(Type F_4). There exists an open dense subset $\mathscr{S}$ of $(y + V) - D$ and hence of $y + V$, which is an affine slice to coadjoint orbits in $\mathfrak{b}_E^*$.*

Proof. We start with y being defined by Π given in 11.3 and V by $\{sg(i)\theta(\alpha_i)\}_{i=1}^4 = \{-2342, 0100, -1242, 0122\}$. Applying the exotic $i = 1$ switch replaces y by $y' := s_{\alpha_1} y$ which is given by $\{1342, -1232, 1121, -1220\}$ and V by V' given by $\Upsilon := \{2342, 1100, -1242, 1122\}$.

By 9.3 the evaluation of $Y(\mathfrak{b}_E)$ on $y' + V'$ contains the (subset) of generators $\{x_i := x_{\upsilon_i}: i = 1, 2, 4\}$ of $R[y' + V']$. It is clear that this is also true of its restriction to $y'' + V''$. Set (by a slight abuse of notation) $x_3 = x_{\upsilon_3}$ with $\upsilon_3 = 0121$. It remains to show that the generator $a_{2\varpi_3}$ of weight $2\varpi_3$ and degree 4 has image $x_1 x_3 + 1$.

Set $\beta_1 = 1342$, $\beta_3 = 1121$, which form the set defining y''. Observe that $\beta_1 + \beta_2 3 + \upsilon_1 + \upsilon_3 = 2\varpi_3$. Set $x_i' = x_{\beta_i}$, $y_i' = \varsigma(x_{\beta_i})$: $i = 1, 3$ and $y_i = \varsigma(x_i)$: $i = 1, 2, 3, 4$. We show that $y_1 y_3' y_1' y_3 v_{\varpi_3}$ is non-zero in the $\mathfrak{g}$ module $V(\varpi_3)$, hence a non-zero multiple of $v_{-\varpi_3}$. This gives the contribution $x_1 x_3$ to the image of $a_{2\varpi_3}$. The contribution of 1 comes from the non-vanishing of $y_1'^2 y_3'^2 v_{\varpi_3}$, which is a trivial consequence of orthogonality.

Our assertion follows from standard $\mathfrak{sl}(2)$ theory. First, $(\upsilon_3, \varpi_3) = 1$ and $(\upsilon_3, \upsilon_3) = 1$, so $y_3^2 v_{\varpi_3} \neq 0$. Second, $(\beta_1, \varpi_3 - \upsilon_3) = 2 - 1$, so $y_1' y_3 v_{\varpi_3} \neq 0$. Third, $(\beta_2, \varpi_3 - \upsilon_3 - \beta_1) = 1$, so $y_3' y_1' y_3 v_{\varpi_3} \neq 0$. Finally $(\upsilon_1, \varpi_3 - \upsilon_3 - \beta_1 - \beta_2) = 2 - 1 - 1$, whilst $x_1 y_3' y_1' y_3 v_{\varpi_3} = y_3^2 \varpi_3 \neq 0$ and so $y_1 y_3' y_1' y_3 v_{\varpi_3} \neq 0$. $\square$

10.3. Retain the above notation and conventions but now with $\mathfrak{g}$ of type B_{2n}.

Define y'' by the subset $\{\varepsilon_1 + \varepsilon_3, \varepsilon_{2i} + \varepsilon_{2i+3}\colon\ i = 1, 2, \dots, n-2, \varepsilon_{2n-2} + \varepsilon_{2n}\}$ and V'' by the subset $\{\varepsilon_1 - \varepsilon_3, \varepsilon_{2i} - \varepsilon_{2i+3}\colon\ i = 1, 2, \dots, n-2, \varepsilon_{2n-2} - \varepsilon_{2n}, \varepsilon_1 + \varepsilon_2, \varepsilon_{2i} + \varepsilon_{2i+2}\colon\ i = 1, 2, \dots, n-2, \varepsilon_3 + \varepsilon_5\}$.

Notice that $\varepsilon_1 + \varepsilon_2$ is the highest root β_*, let D denote the zero locus of x_{β_*} in $y'' + V''$.

Lemma *(Type B_{2n}). There exists an open dense subset $\mathscr{S}$ of $(y'' + V'') - D$ and hence of $y'' + V''$ which is an affine slice to coadjoint orbits in $\mathfrak{b}_E^*$.*

Proof. We start with y being defined by Π given in 11.3 and V by $\{sg(i)\theta(\alpha_i)\}_{i=1}^{2n} = \{(\varepsilon_{2i-1} - \varepsilon_{2i}), -(\varepsilon_1 + \varepsilon_{2i})\colon i = 1, 2, \dots, n\}$.

Following 9.4 we set $s' := \prod_{i=1}^{n-1} s_{2i}$ and $y' := s'y$ which is given by $\{-\varepsilon_1, -(\varepsilon_{2i} + \varepsilon_{2i+1})\colon i = 1, 2, \dots, n-1, \varepsilon_1 + \varepsilon_3, \varepsilon_{2i} + \varepsilon_{2i+3}\colon i = 1, 2, \dots, n-2, \varepsilon_{2n-2} + \varepsilon_{2n}\}$.

Again following 9.4 we take V' to be given by $\{\varepsilon_1 - \varepsilon_3, \varepsilon_{2i} - \varepsilon_{2i+3}\colon i = 1, 2, \dots, n-2, \varepsilon_{2n-2} - \varepsilon_{2n}, \varepsilon_1 + \varepsilon_2, \varepsilon_{2i} + \varepsilon_{2i+2}\colon i = 1, 2, \dots, n-2, -(\varepsilon_1 + \varepsilon_{2n})\}$.

By 9.3 the evaluation of $Y(\mathfrak{b}_E)$ on $y' + V'$ has image containing all the generators of $R[y' + V']$ excepting $x_{-(\varepsilon_1+\varepsilon_{2n})}$. It is clear that this is also true of its restriction to $y'' + V''$.

Observe that V'' is obtained from V' by replacing $-(\varepsilon_1 + \varepsilon_{2n})\}$ by $\varepsilon_3 + \varepsilon_5$. Thus it remains to show that the image of $a_{2\varpi_{2n}}$ (which has weight $2\varpi_{2n} = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{2n}$ and degree n) obtained by restriction to $y'' + V''$ is just $x_{\beta_*}x_{\varepsilon_3+\varepsilon_5} + 1$.

For the above we first observe that $2\varpi_{2n} = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{2n}$, can be written as the sum of $n-2$ terms from the set defining y'', namely $\{\varepsilon_{2i} + \varepsilon_{2i+3}\colon i = 2, 3, \dots, n-2, \varepsilon_{2n-2} + \varepsilon_{2n}\}$ and either $\{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_5\}$ coming from the set defining V'', or $\{\varepsilon_1 + \varepsilon_3, \varepsilon_2 + \varepsilon_5\}$, coming from the set defining y''. Both sums are orthogonal ones, so we obtain from the first (resp. second) the contribution $x_{\beta_*}x_{\varepsilon_3+\varepsilon_5}$ (resp. 1) to the image of $a_{2\varpi_{2n}}$, by evaluation of the matrix element $b_{\xi_{-\varpi_{2n}}, v_{\varpi_{2n}}}$ on the corresponding products of root vectors (as explained several times previously, for example in 3.4). □

11. Appendix

The following two sections give some additional information which our proofs do not require; but which help to explain their nature.

11.1. We have the following complement to Lemma 5.7. Recall the definition of r in 4.4.

Lemma. *Assume $\kappa(\alpha) = \alpha$. Orthogonality fails if and only if one of the following hold*

(i) $h_*(\alpha) = 0$, *Δ is not simply-laced and the coefficient of α in $\theta(\alpha)$, is > 2.*
(ii) $h_*(\alpha) = -1$, *Δ is simply-laced and the coefficient of α in some $|\gamma|\colon \gamma \in \Pi_{\tau(\alpha)}$, is odd and > 1.*
(iii) $h_*(\alpha) = -1$, *Δ is not simply-laced, the coefficient of α in some $|\gamma|\colon \gamma \in \Pi_{\tau(\alpha)}$ long, is odd and > 1, or the coefficient of α in some $|\gamma|\colon \gamma \in \Pi_{\tau(\alpha)}$ short, is $> r$.*
(iv) *In all cases orthogonality failure implies that the coefficient of α in $\theta(\alpha)$ is > 2.*

Proof. By virtue of 4.6 and the orthogonality of the elements of $\Pi_{\tau(\alpha)}$, orthogonality failure exactly occurs when a given $\gamma \in \Pi_{\tau(\alpha)}$ occurs in both in $\theta(\alpha)$ and in the remaining terms in ϖ_α. Taking account of the definition of θ, the above conditions are easily be seen to express this possibility. Nevertheless we spell out the details. The hypothesis of the lemma implies that c is even.

The hypotheses of the lemma and (i) imply that Δ is not simply-laced. Moreover α is a short root as is also the unique root γ in Claim (1) of 4.5. Then orthogonality failure exactly occurs when the coefficient of α in $|\gamma|$ is > 1 and so the coefficient in their sum $\theta(\alpha)$ is > 2.

For (ii), the $\gamma \in \Pi_{\tau(\alpha)}$ which occur in $\theta(\alpha)$ have an odd coefficient by Claim (2) of 4.5 and can only occur in the remaining terms of ϖ_α, if this odd coefficient is > 1. The proof of (iii) similarly uses the assertions in Claim (2). The proof of (iv) is simply by summing coefficients. □

11.2. Although the conditions of Lemmas 5.7, 11.1 may seem rather complicated, the extraordinary thing is that there are almost never satisfied. Indeed (iv) of Lemma 11.1 already forces Δ to be of exceptional type as so does the conclusion of Lemma 5.7.

To the above we may add that if Δ admits just one simple root α such that $h_*(\alpha) = -1$, then $\theta(\alpha)$ is the unique highest root β_* and moreover the latter equals ϖ_α. Apart from this case orthogonality failure occurs in at most the following cases (described using the Bourbaki [5] notation).

Type E_6, when $\alpha = \alpha_3$ or its κ transform α_5,
Type F_4, when $\alpha = \alpha_3$,
Type E_7, when $\alpha = \alpha_4$,
Type E_8, when $\alpha = \alpha_i$: $i = 1, 4, 6$.

To show that orthogonality really does fail we must inspect the set $\Pi_{\tau(\alpha)}$ to see if the criteria of Lemmas 11.1, 11.2 apply. In fact all these cases exhibit orthogonality failure.

We remark that in the last three cases and even in type F_4 when $\varepsilon_\alpha = 1$ one cannot find a good ordering (cf. 5.6).

11.3. Recall 2.21, the definition of Π. At least for $\mathfrak{g}$ classical the simplest way we found to compute Π is to use the truth of Theorem 3.2, first computing $h \in \mathfrak{h}$ defined there. We only need these details for (5) of 9.2 and possibly this too could be avoided.

Below we list for all types the value of h and the resulting elements of Δ on which h takes the value 2. By Theorem 3.2, this set is just Π. We use the Bourbaki [5] notation, using the most convenient convention in each case. Recall our convention in 2.15, concerning the choice of π_a. Indeed for c even, there is a unique simple root α_* conjugate to the highest root β_* and we chose $\alpha_* \in \pi_a$. For $c = 2m + 1$, that is in type A_{2m}, we chose $I_a = \{2, 4, \dots, 2m\}$. For the degrees and the weights of the generators of $Y(\mathfrak{n})$ one must consult [18, Tables I, II] using the corrections in [12, Table].

We first describe all classical cases for which $c = 2m$. Recall 2.15 our convention that for c even we choose $\alpha_* \in \pi_a$.

A_{2m-1}.

In this case α_* is the central simple root, that is α_m, so $m \in I_a$. Thus h is the unique κ invariant element of $\mathfrak{h}$ satisfying

$$h(\varpi_i) = (-1)^{m-(i-1)} i, \quad \forall i = 1, 2, \dots, m + 1.$$

From this one checks that

$$h = (-1)^m \sum_{i=1}^{m} (-1)^{i-1} (2i - 1)(\varepsilon_i - \varepsilon_{2m+1-i}).$$

Thus

$$\Pi = (-1)^m \bigl\{\varepsilon_1 - \varepsilon_{2m}, (-1)^i (\varepsilon_i - \varepsilon_{2m-i}, \varepsilon_{i+1} - \varepsilon_{2m+1-i}) \colon i = 1, 2, \dots, m - 1\bigr\}.$$

B_m.

In this case α_* is the unique long simple root adjacent to a short simple root, that is α_{m-1}, so $m \in I_a$. Inserting the degrees and weights of generators from [12, Table] into 3.2 one obtains $h(\varepsilon_1+, \dots, +\varepsilon_i) = 2(-1)^{m-i}[(i+1)/2]$, $\forall i$ and so

$$h = 2(-1)^m \sum_{i=1}^{m} (-1)^i i \varepsilon_i.$$

Thus

$$\Pi = (-1)^m \bigl\{-\varepsilon_1, (-1)^{i-1}(\varepsilon_i + \varepsilon_{i+1}) \colon i = 1, 2, \dots, m-1\bigr\}.$$

C_m.

In this case α_* is the unique long simple root adjacent to a short simple root, that is α_m, so $m \in I_a$. Inserting the degrees and weights of the generators from [18, Table I] into 3.2 one obtains $h(\varepsilon_1+, \cdots, +\varepsilon_i) = 2(-1)^{m-(i-1)} i$, $\forall i$ and so

$$h = (-1)^m \sum_{i=1}^{m} (-1)^{i-1}(2i-1)\varepsilon_i.$$

Thus

$$\Pi = (-1)^m \bigl\{2\varepsilon_1, (-1)^i(\varepsilon_i + \varepsilon_{i+1}) \colon i = 1, 2, \dots, m-1\bigr\}.$$

D_{m+1}.

In this case α_* is the unique trivalent simple root, that α_{m-1}, so $m-1 \in I_a$. Then using [12, Table] as above one obtains

$$h = 2(-1)^m \sum_{i=1}^{m} (-1)^i i \varepsilon_i.$$

Thus

$$\Pi = (-1)^m \bigl\{(-1)^{i-1}(\varepsilon_i + \varepsilon_{i+1}) \colon i = 1, 2, \dots, m-1, -(\varepsilon_1 \pm \varepsilon_{m+1})\bigr\}.$$

The remaining cases for which c is even are all exceptional. They are described below.

Type E_6.

In this case $c = 2m$ with $m = 6$, the unique trivalent simple root being α_4. We describe h, computed through [18, Table I], as a sum of coweights (resp. coroots) by displaying their coefficients over the corresponding node in the Dynkin diagram below on the left (resp. right). Through [18, Table I] we obtain

$$\begin{matrix} -2 & 4 & -6 & 4 & -2 \\ & & 2 & & \end{matrix}, \qquad \begin{matrix} -8 & 16 & -22 & 16 & -8 \\ & & -10 & & \end{matrix}.$$

Thus

$$\varPi_a = \left\{ \begin{matrix} 1 & 1 & 1 & 1 & 0 \\ & & 0 & & \end{matrix}, \begin{matrix} 0 & 1 & 1 & 1 & 1 \\ & & 0 & & \end{matrix}, \begin{matrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \end{matrix} \right\},$$

$$\varPi_b = -\left\{ \begin{matrix} 0 & 1 & 2 & 1 & 0 \\ & & 1 & & \end{matrix}, \begin{matrix} 1 & 1 & 2 & 2 & 1 \\ & & 1 & & \end{matrix}, \begin{matrix} 1 & 2 & 2 & 1 & 1 \\ & & 1 & & \end{matrix} \right\}.$$

Type E_7.

In this case $c = 2m$ with $m = 9$, the unique trivalent simple root being α_4. We describe h, computed through [12, Table], as a sum of coweights (resp. coroots) by displaying their coefficients over the corresponding node in the Dynkin diagram below on the left (resp. right).

$$\begin{matrix} -4 & 12 & -16 & 14 & -8 & 6 \\ & & 10r & & & \end{matrix}, \qquad \begin{matrix} -10 & 22 & -34 & 26 & -18 & 10 \\ & & 18 & & & \end{matrix},$$

$$\varPi_a = -\left\{ \begin{matrix} 0 & 1 & 2 & 1 & 0 & 0 \\ & & 1 & & & \end{matrix}, \begin{matrix} 0 & 1 & 2 & 2 & 2 & 1 \\ & & 1 & & & \end{matrix}, \begin{matrix} 2 & 3 & 4 & 3 & 2 & 1 \\ & & 2 & & & \end{matrix} \right\}.$$

Thus

$$\varPi_b = \left\{ \begin{matrix} 0 & 0 & 1 & 1 & 1 & 1 \\ & & 1 & & & \end{matrix}, \begin{matrix} 1 & 2 & 3 & 2 & 1 & 0 \\ & & 2 & & & \end{matrix}, \begin{matrix} 1 & 2 & 2 & 1 & 1 & 1 \\ & & 1 & & & \end{matrix}, \begin{matrix} 1 & 2 & 3 & 3 & 2 & 1 \\ & & 1 & & & \end{matrix} \right\}.$$

Type E_8.

In this case $c = 2m$ with $m = 15$, the unique trivalent simple root being α_4. We describe h, computed through [12, Table], as a sum of coweights (resp. coroots) by displaying their coefficients over the corresponding node in the Dynkin diagram below on the left (resp. right).

$$\begin{matrix} -4 & 10 & -14 & 12 & -8 & 6 & -2 \\ & & 8 & & & & \end{matrix}, \qquad \begin{matrix} -18 & 38 & -58 & 46 & -34 & 22 & -10 \\ & & 30 & & & & \end{matrix}.$$

Thus

$$\varPi_a = -\left\{ \begin{matrix} 0 & 1 & 2 & 2 & 2 & 1 & 0 \\ & & 1 & & & & \end{matrix}, \begin{matrix} 2 & 4 & 6 & 5 & 4 & 3 & 2 \\ & & 3 & & & & \end{matrix}, \begin{matrix} 2 & 3 & 4 & 3 & 2 & 1 & 0 \\ & & 2 & & & & \end{matrix}, \begin{matrix} 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ & & 1 & & & & \end{matrix} \right\},$$

$$\varPi_b = \left\{ \begin{matrix} 1 & 2 & 4 & 4 & 3 & 2 & 1 \\ & & 2 & & & & \end{matrix}, \begin{matrix} 1 & 2 & 3 & 2 & 2 & 2 & 1 \\ & & 2 & & & & \end{matrix}, \begin{matrix} 2 & 3 & 5 & 4 & 3 & 2 & 1 \\ & & 2 & & & & \end{matrix}, \begin{matrix} 1 & 2 & 3 & 2 & 1 & 0 & 0 \\ & & 1 & & & & \end{matrix} \right\}.$$

Type F_4.

In this case $c = 2m$ with $m = 6$, and α_* being the unique long simple root adjacent to a short simple root, that α_2. Using [18, Table I], we obtain

$$h = 2\varepsilon_1 + -4\varepsilon_2 + 6\varepsilon_3 - 16\varepsilon_4.$$

Thus

$$\varPi = \{2342, -1232, 0121, -1220\}.$$

Type G_2.

In this case $c = 2m$ with $m = 3$, and α_* being the unique long simple root, that is α_2. Using [18, Table I], we obtain

$$h(\varpi_1) = 2, \qquad h(\varpi_2) = -2.$$

Thus

$$\Pi = \{-(3\alpha_1 + 2\alpha_2), (2\alpha_1 + \alpha_2)\}.$$

Type A_{2m}.
In this case $c = 2m + 1$. Using [18, Table I], we obtain

$$h(\varpi_i + \varpi_{2n+1-i}) = 2i: \quad i = 1, 2, \dots, n.$$

It follows in particular that $h(\beta) = 2$, $\forall \beta \in \mathscr{K}$. Now $\Pi_b \subset \Delta^+$ and has the same cardinality as $\mathscr{K}$. Consequently $\Pi_b = \mathscr{K}$.

On the other hand by Lemma 2.8(ii) one has $w_0 = \sigma^{(2m+1)}$. Consequently $\Pi_a = \sigma^{(m)}\pi_a = -\sigma^{(m)} w_0 \pi_b = -\sigma^{(m+1)}\pi_b = -\sigma_{\kappa^m(a)}\mathscr{K}$, so then $\Pi_a = -\sigma_{\kappa^m(a)}\mathscr{K}$.

11.4. Examples of slices for the coadjoint action of the truncated Borel.

Example 1. Take $\pi = \{\alpha_1, \alpha_2\}$ of type A_2. Set $\beta = \alpha_1 + \alpha_2$, which is the highest root and choose $h \in \mathfrak{h}$, zero on β. Then up to suitable scaling, the generators of $Y(\mathfrak{b}_E)$ are $x_\beta, hx_\beta + x_{\alpha_1}x_{\alpha_2}$. Set $y = x_{-\alpha_2}$, $V = \mathbb{C}x_{-\beta} + \mathbb{C}x_{-\alpha_1}$. One easily checks that the restriction induces an isomorphism of $Y(\mathfrak{b}_E)$ onto $R[y + V]$. Thus $y + V$ is an algebraic slice. Again (h, y) is an adapted pair in the sense of 7.12. Thus $y + V$ is an affine slice consisting of just regular elements. One checks that $\mathfrak{b}_E^* - B_E(y + V) = \mathbb{C}h + \mathbb{C}x_{-\alpha_1}$, which is of codimension 2 and in agreement with the smoothness of the morphism $B_E \times (y + V) \to B_E(y + V)$, and the remarks in 7.2. One may note that this closed set consists of the regular B_E orbit containing $x_{-\alpha_1}$ and the set $\mathbb{C}h$ which consists of trivial orbits.

Notice we can replace y by $x_{-\alpha_2} + x_\beta$, since x_β vanishes on $\mathfrak{b}_E$. It is a regular nilpotent element of $\mathfrak{g}$. It is the choice prescribed by Theorem 6.2, for the construction of a metaslice, which in this special case is already an algebraic slice.

Example 2. Take $\pi = \{\alpha_1, \alpha_2\}$ of type C_2 with α_2 long. Set $\beta = 2\alpha_1 + \alpha_2$, which is the highest root. One has $\mathfrak{b}_E = \mathfrak{n}$. Up to suitable scaling, the generators of $Y(\mathfrak{n})$ are $x_\beta, x_{\alpha_2}x_\beta + x^2_{\alpha_1+\alpha_2}$. Since x_β is already linear then V must contain the direct summand $\mathbb{C}x_{-\beta}$, which leaves us with the quadratic generator. Moreover there is nothing to be gained by having y to be non-zero on x_{α_2}. Thus we are left with a square and so there is no way to obtain an algebraic slice to the coadjoint action in the sense of 7.4. However notice that if we set $y = x_\beta + x_{-(\alpha_1+\alpha_2)}$, $V = \mathbb{C}x_{-\beta} + \mathbb{C}x_{\alpha_2}$, then this quadratic element is reduced to scalars on $y + V$, whilst its κ translate restricts to the linear function $x_{-\alpha_2}$. Thus $y + V$ is a metaslice in the sense of 1.10. This was the prescribed solution given in 6.2.

On the other hand we can take $y = 0$ and $V = \mathbb{C}x_{-\alpha_2} + \mathbb{C}x_{-\beta}$. Then the image of the restriction map of $Y(\mathfrak{n})$ into $R[V]$ is the subalgebra generated by $x_\beta, x_\beta x_{\alpha_2}$, which coincides with $R[V]$ after inverting the highest root vector x_β. Let D be the zero locus of x_β in V. It follows that $Y(\mathfrak{n})$ separates the points of $V - D$. One may check for all $s \in V - D$ that $T_{s,Ns} = \mathbb{C}x_{-(\alpha_1+\alpha_2)} + \mathbb{C}x_{\alpha_1}$, which therefore has null intersection with $T_{s,V} = V$. Thus $V - D$ is an affine slice to the coadjoint action of $B_E = N$.

One may check that $\{ax_{-\alpha_1} + bx_{-(\alpha_1+\alpha_2)} + cx_{-\alpha_2}\colon a, b, c \in \mathbb{C} \mid (a, b) \neq (0, 0)\} = \mathfrak{n}^* - AV$. This has codimension 1 in $\mathfrak{n}^*$; but is not closed. On the other hand D consists of exactly the non-regular (and indeed trivial) orbits meeting V. Moreover $\mathbb{C}x_{-\alpha_1} + \mathbb{C}x_{-(\alpha_1+\alpha_2)} + \mathbb{C}x_{-\alpha_2} = \mathfrak{n}^* - A(V - D)$, is closed, that is $A(V - D)$ is open in $\mathfrak{n}$. This was to be expected from the fact that $A \times (V - C) \longrightarrow \mathfrak{n}$ is a smooth morphism.

Example 3. Take $\pi = \{\alpha_1, \alpha_2\}$ of type G_2 with α_2 long. Set $\beta = 3\alpha_1 + 2\alpha_2$, which is the highest root. One has $\mathfrak{b}_E = \mathfrak{n}$. Up to suitable scaling, the generators of $Y(\mathfrak{n})$ are x_β, $x_{\alpha_1}x_\beta + x^2_{2\alpha_1+\alpha_2} + x_{\alpha_1+\alpha_2}x_{3\alpha_1+\alpha_2}$. Since x_β is already linear then V must contain the direct summand $\mathbb{C}x_\beta$, which leaves us with the quadratic generator. Moreover there is nothing to be gained by having y to be non-zero on x_{α_1}. However unlike type C_2 we are not just left with a square. If we set $y = x_{-\beta} + x_{2\alpha_1+\alpha_2}$, $V = \mathbb{C}x_\beta + \mathbb{C}x_{-\alpha_1}$, then this quadratic element restricts to the linear function x_{α_1} on $y + V$, whilst its κ translate reduces to scalars. Thus $y + V$ is a metaslice in the sense of 6.2.

On the other hand in contrast to type C_2, we have the alternative of setting $y' = s_2 y = x_{-(3\alpha_1+\alpha_2)} + x_{2\alpha_1+\alpha_2}$, $V = \mathbb{C}x_{-\beta} + \mathbb{C}x_{-(\alpha_1+\alpha_2)}$ as prescribed by exotic switching (9.3). This gives an algebraic slice to the coadjoint action of $\mathfrak{n}$ in the sense of 7.4. As a linear function on $\mathfrak{n}$ we may drop the second term in y.

Using for example that the $\mathfrak{h}$ stable complement to $\mathbb{C}x_{\alpha_1}$ is a Heisenberg algebra one checks that $\mathfrak{n}^*_{reg}$ consists of all elements non-vanishing on x_β, that is having a non-zero coefficient of $x_{-\beta}$, and consists of four-dimensional N orbits. In particular $(y + V)_{reg} = y + \mathbb{C}^* x_{-\beta} + \mathbb{C}x_{-(\alpha_1+\alpha_2)}$. One then checks that $N(y + V)_{reg} = \mathfrak{n}^*_{reg}$, which is of course open in $\mathfrak{n}^*$. Set $(y + V)_s = y + \mathbb{C}x_{-(\alpha_1+\alpha_2)}$. It is the complement of $(y + V)_{reg}$ in $y + V$ and consists of two-dimensional N orbits. Let D denote the zero locus of $x_{3\alpha_1+\alpha_2}x_{\alpha_2} - x_{2\alpha_1+\alpha_2}x_{\alpha_1+\alpha_2}$ in $y + \mathbb{C}x{-}(2\alpha_1 + \alpha_2) + \mathbb{C}x{-}(\alpha_1 + \alpha_2) + \mathbb{C}x_{-\alpha_2}$. One checks that $N(y + V)_s = \mathbb{C}x_{-\alpha_1} \times D$, which is of course closed and three-dimensional. We conclude that $N(y + V)$ has codimension 1 in $\mathfrak{n}^*$, but is not open.

Example 4. Let $\mathfrak{a}(n)$ denote the standard filiform Lie algebra of dimension n, defined to have basis $\{x_i\}_{i=1}^n$ with only non-zero brackets being $[x_1, x_i] = x_{i+1}$: $i = 2, \dots, n - 1$. For $n = 3$ this is just the Heisenberg Lie algebra, for $n = 4$ it is the nilradical of the Borel in type C_2. Set $\mathfrak{a} = \mathfrak{a}(5)$. It was first noted by Dixmier that $Y(\mathfrak{a})$ is not polynomial (see [31, Introduction] for discussion and further references). Trivially $z_1 := x_5 \in Y(\mathfrak{a})$ and one easily checks that $Y(\mathfrak{a})[z_1^{-1} = \mathbb{C}[z_1, z_1^{-1}, z_2, z_3]$, where

$$z_2 = x_3x_5 - \frac{1}{2}x_4^2, \qquad z_3 = x_2x_5^2 + \frac{1}{3}x_4^3 - x_3x_4x_5.$$

On the other hand $9z_3^2 - 8z_2^3$ is divisible by x_5 and therefore $Y(\mathfrak{a})$ cannot be polynomial.

Let $\{y_i\}_{i=1}^5$ be a dual basis for $\mathfrak{a}^*$. Set $V = \mathbb{C}y_2 + \mathbb{C}y_3 + \mathbb{C}y_5$. Let D be the zero locus of x_5 in V and set $\mathscr{S} = V - D$. Obviously the above three generators separate points of $\mathscr{S}$. One easily checks that $\mathscr{S} = \mathscr{S}_{reg}$ and that the $T_{y,Ay} \oplus T_{y,V} = T_{y,\mathfrak{a}^*}$. Thus $\mathscr{S}$ is an affine slice to coadjoint action. Again one checks that $A\mathscr{S}_{reg}$ is just the zero locus of z_1, in keeping with (though not implied by) Proposition 7.4. Thus the embedding of $Y(\mathfrak{a})$ into $R[V]$ defined by restriction of functions induces an isomorphism of $Y(\mathfrak{a})[z_1^{-1}]$ onto $R[V][z_1^{-1}]$, as may be easily checked through the formulae above, but is not itself onto as expected by $Y(\mathfrak{a})$ not being polynomial. Of course as in Example 1, the image could be proper yet polynomial. Here it is easy to check that this is not true.

12. Supplementary appendix – a result of Hinich

12.1. Let $\mathbf{k}$ be a field and $\iota : R \hookrightarrow S$, a faithfully flat embedding of commutative $\mathbf{k}$-algebras with S reduced and finitely generated as a $\mathbf{k}$-algebra. Given I an ideal of S let $\sqrt{I}$ denote its radical that is the intersection of the maximal ideals containing I. The following was reconstructed from discussions with V. Hinich. The main point of the proof is to adapt a trick of S. Amitsur.

Lemma. *Take $s \in S$ such that $s \in \mathbf{k} + \sqrt{\mathbf{m}S}$, for all $\mathbf{m} \in Max\, R$. Then $s \in R$.*

Proof. Let $C \hookrightarrow D$ be an embedding of commutative rings with D finitely generated.

Let $\mathbf{n}$ be a maximal ideal of D. Since D is finitely generated, $D/\mathbf{n}$ is a finite field extension of $\mathbf{k}$. The $\mathbf{k}$-subalgebra $C/(C \cap \mathbf{n})$ is again a field so $C \cap \mathbf{n} \in Max\, C$. Again the radical of D, namely $\sqrt{0}$, is a nilpotent ideal by the nullstellensatz.

By flatness ι induces an algebra embedding $1 \otimes \iota$ of $S = R \otimes_R S$ into $T := S \otimes_R S$. Then $(1 \otimes \iota)\iota$ is an embedding of R into T. (We identify R (resp. S) with its image in T.) Moreover T is a quotient of $S \otimes S$, hence finitely generated by the hypothesis on S. Let N denote its radical. It is nilpotent by the first paragraph of the proof.

Now take $s \in S$ satisfying the hypothesis of the lemma. Take $\mathbf{n} \in Max\, T$ and set $\mathbf{m} := \mathbf{n} \cap R \in Max\, R$. Then $\sqrt{\mathbf{m}S} \subset \mathbf{n}$.

By the hypothesis on s there exist $c \in \mathbf{k}$ and $n \in \mathbb{N}^+$ such that $(s - c)^n \in \mathbf{m}S$. Then $(1 \otimes (s - c))^n \in S \otimes_R \mathbf{m}S$ and so $1 \otimes (s - c) \in \mathbf{n}$. On the other hand $((s - c) \otimes 1)^n \in \mathbf{m}S \otimes S = S \otimes_R \mathbf{m}S$ and so again $(s - c) \otimes 1 \in \mathbf{n}$. Subtracting we conclude that $s \otimes 1 - 1 \otimes s \in \mathbf{n}$. Since $\mathbf{n}$ is arbitrary we conclude by that

$$s \otimes 1 - 1 \otimes s \in N. \tag{$*$}$$

Consider R, Rs, N as R submodules of T. By$(*)$ we obtain $(R + N + Rs) \otimes_R S = (R + N) \otimes_R S$. Thus the quotient $(R + N + Rs)/(R + N) \otimes_R S$ is zero. By faithfulness this implies $(R + N + Rs)/(R + N) = 0$. Hence $s \in R + N$. Since $R \subset S$ and $S \cap N = 0$, by the hypothesis that S is reduced, we obtain $s \in R$, as required. □

Remarks. (1) A crucial point is that $\mathbf{m}S \otimes_R S = S \otimes_R \mathbf{m}S$.

(2) Flatness is needed because the image of S in T might not otherwise be reduced. Faithfulness is also necessary. For example the embedding $\mathbb{Z}$ into $\mathbb{Q}$ is flat, but not faithful. Now in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ we have for any positive integer n that $1/n \otimes 1 = 1/n \otimes n/n = 1 \otimes 1/n$, yet $1/n \notin \mathbb{Z}$ if $n > 1$.

12.2. Let $\varphi : Y \to Z$ be a smooth surjective morphism of algebraic varieties and set $R = R[Z]$, $S = R[Y]$. Then S is finitely generated and reduced. Moreover (cf. 7.2) the comorphism $\varphi^* : R \to S$ is injective and faithfully flat. Let S^φ denote the set of elements of S constant on the fibres of φ, that $S^\varphi = \{s \in S \mid s(y) = s(y'),\ \text{if } \varphi(y) = \varphi(y')\}$. It is clear that $\varphi^*(R) \subset S^\varphi$. Conversely suppose $s \in S^\varphi$ and let $\mathbf{m}_z$ denote the maximal ideal corresponding to $z \in Z$. Let c be the constant value of s on $\varphi^{-1}(z)$. Then $s - c$ belongs to the ideal I of definition of $\overline{\varphi^{-1}(z)}$. One has $I = \{s \in S \mid s(\varphi^{-1}z) = 0\}$. Since $\varphi^*(\mathbf{m}_z)(\varphi^{-1}yz) = \mathbf{m}_z(\varphi\varphi^{-1}) = 0$, we obtain $I \supset S\varphi^*(\mathbf{m}_z)$. Conversely one checks that the associated variety of $S\varphi^*(\mathbf{m}_z)$ is $\varphi^{-1}(z)$. Through the nullstellensatz we obtain $I = \sqrt{S\varphi^*(\mathbf{m}_z)}$ and this holds for all $z \in Z$. Thus through 12.2 we obtain

Corollary. *The comorphism φ^* induces an isomorphism of R onto S^φ.*

12.3. We apply the previous result to the situation described in 7.3 with $Y = A \otimes \mathscr{S}_{reg}$, $Z = A\mathscr{S}_{reg}$ with the morphism φ defined by the action of A, which is of course surjective. Moreover under the hypotheses of 7.3, φ is a smooth map. Applying 12.2 leads to the following result of Hinich in which we adopt the conventions of 7.3.

Proposition *(V. Hinich). Let $\mathscr{S}$ be a slice to the action of A on X. Restriction of functions gives an isomorphism of $R[A\mathscr{S}_{reg}]^A$ onto $R[\mathscr{S}_{reg}]$.*

Proof. Set $S = R[A \otimes \mathscr{S}_{reg}]$. Through the hypothesis on $\mathscr{S}$ one has $S^\varphi = \{s \in S \mid s(a, v) = s(a', v),\ \forall a' \in a\mathit{Stab}_A v,\ \forall v \in \mathscr{S}_{reg}\}$. Set $U = (S^\varphi)^A$. By the above $u \in U$ is uniquely determined by its value at the pairs $(e, v) : v \in \mathscr{S}_{reg}$, where e denotes the identity of A and hence by a regular function on $\mathscr{S}_{reg}$, that is U identifies with $R[\mathscr{S}_{reg}]$. On the other hand by 12.2 one has $R[A\mathscr{S}_{reg}]^A \xrightarrow{\sim} U$. Hence the proposition. $\square$

12.4. Hinich offered a second proof of 12.2 which is more geometric. It is also shorter but this is because it needs the theorem of faithfully flat descent [15, Chap. 8]. We sketch very briefly the details.

Recall the notation and hypotheses of 12.2, with Y, Z as in 12.3. Consider the fibre product $Y \times_Z Y$ and p_1 (resp. p_2) the first (resp. second) projection of $Y \times_Z Y$ onto Y. Let $\psi : Y = A \times \mathscr{S}_{reg} \to \mathscr{S}_{reg} \subset Z$, be the projection onto the second factor. Take $a, a' \in A$ and $s, s' \in \mathscr{S}_{reg}$. Then $((a, s), (a', s')) \in Y \times_Z Y$ if and only if $as = a's'$, which by the hypothesis on $\mathscr{S}$ implies $s = s'$. Thus ψp_1 and ψp_2 coincide on $Y \times_Z Y$. Thus any regular function on their common image $\mathscr{S}_{reg}$ gives rise to a regular function f on $Y = A \times \mathscr{S}_{reg}$, whose two inverse images on $Y \times_Z Y$ coincide. Since $\varphi : Y \longrightarrow Z$ is a smooth map, the theorem of faithfully flat descent implies that f is the inverse image of a regular function on Z. This replaces the use of 12.1, 12.2.

Remark. In the notation of 12.2 one may view $T = S \otimes_R S$ as the algebra of regular functions on $Y \times_Z Y$, since smoothness implies that T is reduced [16, 17.5.7]. Again the theorem of faithfully flat descent also uses the Amitsur trick. Thus the two proofs are essentially the same, the first only spelling out more details.

13. Addendum – computation of the fundamental semi-invariant

13.1. Let $\mathfrak{a}$ be a finite dimensional Lie algebra. The fundamental semi-invariant $p_{\mathfrak{a}}$, or simply p, of $\mathfrak{a}$ is defined as follows. Set $\mathfrak{a}^*_s := \mathfrak{a}^* \setminus \mathfrak{a}^*_{reg}$.

Let $\{x_i\}_{i=1}^n$ be a basis for $\mathfrak{a}$ and $\{\xi\}_{i=1}^n$ the dual basis for $\mathfrak{a}^*$. The Poisson bivector

$$P := \sum_{i,j=1}^{n} [x_i, x_j]\xi_i \wedge \xi_j,$$

is an invariant. Moreover $r := \dim \mathfrak{a} - \operatorname{index} \mathfrak{a}$ is even and so we may define $Q := \bigwedge^{r/2} P$. Obviously the coefficients of Q with respect to the Grassmann algebra of $\mathfrak{a}^*$ are homogeneous elements of $S(\mathfrak{a})$ of degree $r/2$. As is well known, $Q(\xi)=0$, exactly when $\xi \in \mathfrak{a}_s^*$. Again Q is the highest power of P which is non-zero.

One defines $p_{\mathfrak{a}}$ to be the greatest common divisor of the coefficients of Q. It is a semi-invariant. It is a scalar if and only $\mathfrak{a}$ is non-singular. Indeed the zero set of $p_{\mathfrak{a}}$ is just the codimension one component of $\mathfrak{a}_s^*$.

13.2. Assume from now on that $S(\mathfrak{a})$ admits no proper semi-invariants and that the invariant algebra $Y(\mathfrak{a})$ is polynomial on $\ell := \operatorname{index}\mathfrak{a}$ generators. Let D be a family of commuting semisimple outer derivations of $\mathfrak{a}$. We can assume that the generators $f_1, f_2, \dots, f_\ell$ are homogeneous and D weight vectors. By [25, Prop. 5.2] the greatest common divisor of $df_1 \wedge df_2 \wedge \dots \wedge df_\ell$ is scalar and so in the notation of [25, 5.4] it becomes ω. Again in the notation of [25, 5.3] we may write $W = p^{-1}Q$ – here W is *not* the Weyl group!

Set $\Omega := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. Given a a D weight vector, let $\operatorname{wt} a$ denote its weight. The following is an immediate consequence of [25, Lemma 5.4], noting that $\operatorname{wt} P = 0$.

Lemma. $\operatorname{wt}\omega = \operatorname{wt}\Omega - \operatorname{wt} p_{\mathfrak{a}}$.

13.3. Now let $\mathfrak{a}$ be a truncated Borel subalgebra $\mathfrak{b}_E$ of a simple Lie algebra $\mathfrak{g}$. We may identify D with the set of outer derivations of $\mathfrak{b}_E$ coming from the complement of the Cartan subalgebra of $\mathfrak{b}_E$ in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then the invariant algebra $Y(\mathfrak{a})$ is polynomial and its generators $a_1, a_2, \dots, a_\ell$: $\ell = rk\,\mathfrak{g}$, can be chosen to be $\mathfrak{h}$ and hence D weight vectors.

Recall Eq. $(*)$ of 3.1. We say that a_i is a missing invariant generator if $\varepsilon_i = \frac{1}{2}$. These are exactly the generators which do not come from the Hopf algebra construction outlined in 1.7. Rather it is their squares that so arise. Let M denote the subset of $I = \{1, 2, \dots, \ell\}$ for which $\varepsilon_i = \frac{1}{2}$.

Proposition. $p_{\mathfrak{b}_E} = \prod_{i\in M} a_i$, *up to a non-zero scalar.*

Proof. Let ρ be the half-sum of the positive roots. It is immediate that $\operatorname{wt}\Omega = 2\rho$. On the other hand by Eq. $(*)$ of 3.1 it is also clear that

$$\sum_{i\in I} \operatorname{wt} a_i^{\frac{1}{\varepsilon_i}} = 2\rho.$$

In view of Lemma 13.2, we conclude that

$$\operatorname{wt} p_{\mathfrak{b}_E} = \operatorname{wt} \prod_{i\in M} a_i =: \mu.$$

On the other hand it follows from equation $(*)$ of that $Y(\mathfrak{b}_E)_\mu$ is one-dimensional. Hence the required assertion. $\square$

13.4. There is another way to describe the fundamental semi-invariant p, namely it is the *square* of the greatest common divisor of the non-vanishing minors of the matrix with entries $\{[x_i, x_j]\}_{i,j=1}^n$. If we could show (for a truncated Borel) that this element obtains by the Hopf algebra construction of 1.7, then we would obtain the conclusion of Proposition 13.3 without having to know the existence of the missing generators. The advantage of such a proof is that it could be expected to pass over to the biparabolic case with a similar conclusion, namely that certain invariants elements obtained from the Hopf dual construction admit square roots, their product being the fundamental semi-invariant of the truncated biparabolic. Actually except for the Borel we do not know of any examples of truncated biparabolics which are non-singular (though this is mainly for want of looking). We remark that by the sum rules in [13] and [25] this can only happen if the upper and lower bounds alluded to in 1.7 do *not* coincide. This can only occur outside types A and C; though there are plenty of cases, but for those we examined the missing invariants (outside the Borel) are not square roots. Yet it would seem certain that such square roots exist, it should just need enough energy to find them.

14. Index of notation

Symbols used frequently are given below in the section where they are first defined.

1.1. $\mathfrak{a}$, A, $\mathfrak{a}^*$, $S(\mathfrak{a})$, $Y(\mathfrak{a})$.
1.4. $\mathfrak{g}$.
1.5. W.
1.6. $\mathfrak{b}$, $\mathfrak{n}$, $Sy(\mathfrak{n})$.
1.7. $U(\mathfrak{g})$, $\mathfrak{p}$.
1.8. $\mathfrak{p}_E$, $\mathfrak{a}^*_{reg}$.
2.1. $\mathfrak{n}^-$, $\mathfrak{h}$, ℓ, Δ, π, $\pi^\vee$, Δ^+, Δ^-, P, P^+, $V(\mu)$, v_μ, s_i, x_α.
2.2. $\pi_a, \pi_b, I_a, I_b, \sigma_a, \sigma_b, \sigma, \widehat{C}, C$.
2.3. w_0.
2.5. π_c, $\pi_c^\vee$, $\mathbf{r}$, $\mathbf{r}_a$, $\mathbf{r}_b$, Δ_c, Δ_c^+, Δ_c^-.
2.6. g_i.
2.7. ψ.
2.13. $w_\subset : c$ even.
2.14. $\beta_*, \alpha_0, \alpha_*, (\ ,\ )$.
2.15. $\Delta_*, \mathscr{K}_\Delta, h_*, h_0$.
2.17. $w_\subset : c$ odd.
2.21. $\varPi$, $\varPi_c$.
3.1. $\varepsilon_i, a_i, \varLambda_0, b_{\xi,v}, b_v, \mathscr{F}, a_v, \deg b$.
3.2. $s(i), h$.
3.3. y.
3.4. $\varPi_a, \varPi_b, y_a, y_b, \varLambda, b_{(i)}, |\gamma|$.
4.2. θ, π_0.
4.3. τ.
4.4. r, ε_α.
4.12. α_c, z.
5.1. $sg(i), I_0, \pi_0, T_0, T_1, T$.
5.4. $b^{op}_{(i)}, \tau^{op}$.
5.12. V_0, V_1, V.
6.1. $b'_{(i)}, b''_{(i)}, b''^{op}_{(i)}$.
6.2. $\varsigma, \varsigma_i, X_i, A(\mathfrak{g})^\varsigma$.
8.8. $A(\mathfrak{g})$.

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References

[1] J. Alev, A. Ooms, M. Van den Bergh, A class of counterexamples to the Gel'fand–Kirillov conjecture, Trans. Amer. Math. Soc. 348 (5) (1996) 1709–1716.

[2] W. Borho, Über Schichten halbeinfacher Lie-Algebren (On sheets of semisimple Lie algebras), Invent. Math. 65 (2) (1981/1982) 283–317 (in German).

[3] W. Borho, H. Kraft, Hanspeter Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen, Comment. Math. Helv. 54 (1) (1979) 61–104 (in German).

[4] W. Borho, R. Rentschler, Sheets and hearts of prime ideals in enveloping algebras of semisimple Lie algebras, J. Algebra 304 (1) (2006) 324–348.

[5] N. Bourbaki, Éléments de mathématique, Groupes et algèbres de Lie, Chapitres 4, 5 et 6 (Elements of Mathematics, Lie Groups and Lie Algebras, Chapters 4, 5 and 6), Masson, Paris, 1981 (in French).

[6] A.J. Coleman, The Betti numbers of the simple Lie groups, Canad. J. Math. 10 (1958) 349–356.

[7] J.-L. Colliot-Theĺeňe, B. Kunyavskii, V.L. Popov, Z. Reichstein, Is the function field of a reductive Lie algebra purely transcendental over the field of invariants for the adjoint action?, arXiv:0901.4358.

[8] J. Dixmier, Sur les représentations unitaires des groupes de Lie nilpotents, IV, Canad. J. Math. 11 (1959) 321–344 (in French).

[9] J. Dixmier, Polarisations dans les algèbres de Lie semi-simples complexes, Bull. Sci. Math. (2) 99 (1) (1975) 45–63.

[10] J. Dixmier, Algèbres enveloppantes (Enveloping Algebras), Reprint of the 1974 original. Les Grands Classiques Gauthier–Villars (Gauthier–Villars Great Classics), Éditions Jacques Gabay, Paris, 1996 (in French).

[11] J. Dixmier, M. Duflo, M. Vergne, Sur la représentation coadjointe d'une algèbre de Lie, Compos. Math. 29 (1974) 309–323 (in French).

[12] F. Fauquant-Millet, A. Joseph, Sur les semi-invariants d'une sous-algèbre parabolique d'une algèbre enveloppante quantifiée (Semi-invariants of a parabolic subalgebra of a quantized enveloping algebra), Transform. Groups 6 (2) (2001) 125–142 (in French).

[13] F. Fauquant-Millet, A. Joseph, Semi-centre de l'algèbre enveloppante d'une sous-algèbre parabolique d'une algèbre de Lie semi-simple, Ann. Sci. École Norm. Sup. 38 (2) (2005) 155–191.

[14] F. Fauquant-Millet, A. Joseph, La somme dans faux degrés—un mystère en théorie des invariants (The sum of the false degrees—A mystery in the theory of invariants), Adv. Math. 217 (4) (2008) 1476–1520 (in French).

[15] A. Grothendieck, Revêtements étales et groupes fondamentales, Lecture Notes in Math., vol. 224, Springer, Berlin, 1971, reprinted in arXiv:math.AG/0206203.

[16] A. Grothendieck, J.A. Dieudonné, Éléments de géométrie algébrique, IV, Publ. Math. Inst. Hautes Etudes Sci. 24 (1965–1967) 32.

[17] C. Hoyt, Private communication.

[18] A. Joseph, A preparation theorem for the prime spectrum of a semisimple Lie algebra, J. Algebra 48 (2) (1977) 241–289.

[19] A. Joseph, Quantum Groups and Their Primitive Ideals, Springer-Verlag, 1995.

[20] A. Joseph, On semi-invariants and index for biparabolic (seaweed) algebras, II, J. Algebra 312 (1) (2007) 158–193.

[21] A. Joseph, Parabolic actions in type A and their eigenslices, Transform. Groups 12 (3) (2007) 515–547.
[22] A. Joseph, Slices for biparabolic coadjoint actions in type A, J. Algebra 319 (12) (2008) 5060–5100.
[23] A. Joseph, Compatible adapted pairs and a common slice theorem for some centralizers, Transform. Groups 13 (3–4) (2008) 637–669.
[24] A. Joseph, A Pentagonal Crystal, the Golden Section, alcove packing and aperiodic tilings, Transfom. Groups 14 (3) (2009) 557–612.
[25] A. Joseph, D. Shafrir, Polynomiality of invariants, unimodularlity and adapted pairs, Transform. Groups 15 (4) (2010) 851–852.
[26] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math. 81 (1959) 973–1032.
[27] B. Kostant, Lie group representations on polynomial rings, Amer. J. Math. 85 (1963) 327–404.
[28] D. Luna, Slices étales, Bull. Soc. Math. France Mémoires 33 (1973) 81–105.
[29] C. Moeglin, Idéaux bilatères des algèbres enveloppantes (Bilateral ideals of enveloping algebras), Bull. Soc. Math. France 108 (2) (1980) 143–186 (in French).
[30] R.V. Moody, J. Patera, Quasicrystals and icosians, J. Phys. A 26 (12) (1993) 2829–2853.
[31] A. Ooms, M. Van den Bergh, A degree inequality for Lie algebras with a regular Poisson semi-center, arXiv:0805.1342.
[32] V.L. Popov, On the "lemma of Seshadri", in: Arithmetic and Geometry of Varieties, Samar. Gos. Univ., Samara, 1992, pp. 133–139 (in Russian), translated in Lie Groups, Their Discrete Subgroups, and Invariant Theory, in: Adv. Soviet Math., vol. 8, Amer. Math. Soc., Providence, RI, 1992, pp. 167–172.
[33] V.L. Popov, Groups, Generators, Syzygies, and Orbits in Invariant Theory, translated from the Russian by A. Martsinkovsky, Transl. Math. Monogr., vol. 100, Amer. Math. Soc., Providence, RI, 1992.
[34] V.L. Popov, Sections in invariant theory, in: The Sophus Lie Memorial Conference, Oslo, 1992, Scand. Univ. Press, Oslo, 1994, pp. 315–361.
[35] M. Rosenlicht, A remark on quotient spaces, An. Acad. Brasil. Cienc. 35 (1963) 487–489.
[36] M. Rosenlicht, Another proof of a theorem on rational cross sections, Pacific J. Math. 20 (1967) 129–133.
[37] W. Schmidt, Die Randwerte homomorpher Funcktionen auf Hermitesch symmetrischen Raumen, Invent. Math. 9 (1969/1970) 61–80 (footnote p. 79).
[38] P. Slodowy, Simple Singularities and Simple Algebraic Groups, Lecture Notes in Math., vol. 815, Springer-Verlag, 1980.
[39] O. Yakimova, A counterexample to Premet's and Joseph's conjectures, Bull. Lond. Math. Soc. 39 (5) (2007) 749–754.